EJECTIONS AND CAPTURES BY SOLAR SYSTEMS

ABSTRACT

This review discusses the circumstances under which bodies may be ejected from, or captured by, solar systems. The review considers theoretical studies.

Section 1 describes the work of Littlewood, who proved in 1953 that otherwise regular systems of gravitating bodies can never eject nor capture a body, with the exception of infinitely rare special cases. This is an unexpected result and goes against the widely held belief that astronomical bodies can capture satellites.

The existence of singularities in Newtonian systems is discussed in Section 2 and it is shown that for a loss or capture to occur, a singularity without a collision must take place.

The connection between singularities and collisions is explored in Section 3 and the work of Painlevé and Saari is summarised to show that in the n-body problem for $n \leq 4$, no systems ever eject or capture bodies except for an infinitely small set in the 4-body case.

Section 4 offers supporting evidence for the existence of noncollision singularities, using the Mather-McGehee model which shows for the 4-body case that collisions can accumulate to eject a body from the system.

The recent work of Xia is discussed in Section 5, including his construction of a noncollision singularity in the 5-body case, and his proof that such constructions can be made for all $n \geq 5$.

The review ends with suggestions for future work (Section 7).

1. INTRODUCTION

In 1953, the mathematician J. E. Littlewood [2, 3] proved an intriguing result. In a short demonstration [1] and subsequent proof [2, 3] Littlewood showed that “a gravitating system of bodies (a generalisation of the Solar System) can never make a capture, even of a speck (or reversely suffer a loss)”. This profound statement requires some clarification, however, as there are limiting cases in which a system can capture or lose a body permanently. Such cases are infinitely rare.

The result of this proof is immediately interesting because it is widely accepted that large bodies like the Sun or Jupiter capture other, minor, astronomical bodies which become their satellites. Littlewood [1] explained that this is still allowed, but that the capture is not permanent and that one of the bodies in the system (not necessarily the captured one) will later be ejected.

In [2], Littlewood defined systems as being regular or not regular. A system of gravitating bodies is defined to be regular for all future time if:

(i) its diameter in real space is bounded for all future time.

(ii) collisions do not occur for all future time.

The same definition can be applied over all past time.

A system cannot be said to have ejected a body until the body is outside the gravitational influence of the system. An ejected body must therefore go to infinity. In this case the system is not regular because its diameter in real space is not bounded. The same applies for capture when a body comes in from outside the gravitational influence of the system, that is, from infinity.

The short demonstration [1] of Littlewood’s proof [2, 3] can be summarised as follows.

If there exists a gravitating system of bodies, one can conjecture that some starting configurations will be regular (as defined above) for all future time, and other starting configurations will not be regular for all future time. In phase space one can form a

set R containing, at a given time *t*, the representative points Q of the regular systems. Then the system is advanced by an arbitrary time increment *dt*. One can then, as above, construct a set R' containing, at time *t* + *dt*, the representative points Q' of the regular systems.
Two points can be noted. Firstly, every point Q in R has a corresponding point Q' in R'. Since every point Q in R is regular from time *t* up to infinity, it will evolve into a point Q' that is regular from time *t* + *dt* up to infinity which thus appears in R'.
Secondly, Liouville's Theorem states that for a dynamical system, the volume of the representative points in phase space remains constant with time. In this case, this means that the volumes of R and R' are identical for any time increment *dt*.

Since the volumes of R and R' are the same, and all the points in R are also in R', the only conclusion is that R and R' contain exactly the same set of configurations. No extra representative points are admitted, nor are any expelled.
Given that the time increment *dt* is arbitrary, this result can be extended into both the infinite past and the infinite future. As a consequence of this the results below follow.

Using the above ideas, Littlewood [2] proved that if every system Q of a set R is regular for all future time, then almost all of the systems Q in R are regular for all past time. The same theorem is true with the times reversed, that is, for all Q in R that are regular for all past time, almost all Q in R are regular for all future time. Almost all, in this context, means all except for an infinitely small set.

Hence if a system does not capture or eject a body or suffer a collision for all future time, then it will not have done so for all past time. This is also true with the times reversed.

These results also have the consequence that if all systems Q in a set R are not regular in the past, then almost all Q in R will not be regular in the future. In other words, systems that have captured a body in the past will eject one in the future, or suffer a collision. Littlewood [1] explicitly stated that the time between these two events is arbitrary, and that it does not have to be the captured body that is ejected. To take this to extremes, the orbital capture by the Sun of a small speck of dust could result in Jupiter being emitted from the solar system.

Littlewood's work also discussed collisions. He knew that the set of systems suffering collisions between two bodies, a simple collision, had measure zero, that is such collisions are infinitely rare. A proof of this was later offered by Saari [10]. It was not however known at the time of Littlewood's work (1953) that the set of systems suffering collisions between three or more bodies, a multiple collision, had measure zero. This is counterintuitive, because it would seem that a multiple collision must be less likely than a simple collision. In 1968, Littlewood [4] left this as an open problem, which was later solved by Saari [10].

2. PROPERTIES OF SINGULARITIES

Littlewood was not the first to consider this problem. The *n*-body problem as an idealized Solar System has been studied for more than a century.

Much of this work has been concentrated on singularities. A singularity is that which stops the system being regular, that is a capture, loss or collision. The study of collisions is important because if a system suffers a collision it is not regular and, from Littlewood's result above, it may then be possible for it to eject or capture a body.

Painlevé [7] provided a useful definition of a singularity. The symbol Δ is used as mathematical shorthand for those configurations where the potential energy is undefined. It is defined in phase space. The potential energy is given by

$$U = \sum_{j<i} \frac{-m_i m_j}{\left|\mathbf{q}_i - \mathbf{q}_j\right|}$$

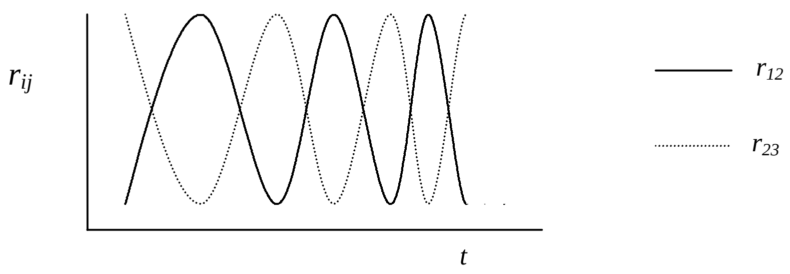
where m_i and m_j are the masses, and $\mathbf{q}_i$ and $\mathbf{q}_j$ the position vectors of the i^{th} and j^{th} bodies.

It is clear that when the denominator goes to zero, that is when two bodies *i* and *j* collide, the potential energy is undefined. All such configurations are on Δ.

Painlevé [7] defined a singularity as:

Definition: A system of gravitating bodies has a singularity at time $t = \sigma$ (where $\sigma < \infty$) if $\boldsymbol{r}(t)$ tends to Δ as t tends to σ, where $\boldsymbol{r}$ is the representative point of the system in phase space.

Painlevé investigated singularities and their causes. It is clear that a collision is a singularity, but this does not imply that all singularities are collisions. He suggested that a form of oscillatory motion, where the separation of different pairs of bodies alternated in r_{min}, the minimum spacing between a pair of bodies, would fit the definition of a singularity but not involve a collision.



r_{ij} represents the distance between bodies i and j.

Figure 1 Painlevé's suggestion for a noncollision singularity.

Figure 2 shows this motion at two times: firstly when bodies 1 and 2 are close and then when bodies 2 and 3 are close. In this scheme, bodies 1 and 3 remain roughly fixed.

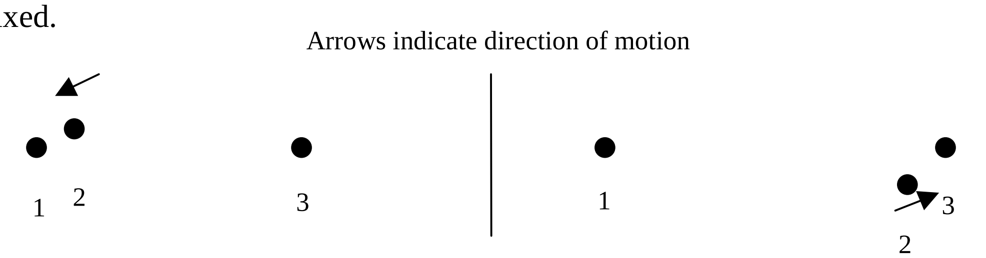


Figure 2 Possible positions of bodies 1, 2 and 3 in Painlevé's proposed oscillatory motion at two different times.

This motion leads to a singularity in that the limit of the minimum spacing approaches zero while the limit superior of the minimum spacing, that is, the maximum value of the minimum spacing, remains positive. To quote Saari and Xia [13], the particles "flirt with colliding without ever doing so".

Such motion, where singularities occur without collisions, leads to the mathematical description of collision and noncollision singularities.

Definition: If as t tends to σ, $\boldsymbol{r}(t)$ tends to a specific point **p** on Δ, then two or more bodies collide and a collision singularity occurs. If there is no such point **p**, then a noncollision singularity occurs.

Although Painlevé's suggested oscillatory motion seems plausible, and would be a noncollision singularity, he was able to show [7] that for the 3-body case, all singularities are collisions, and that oscillatory or any other motion leading to a singularity does not occur. He was unable to extend this result to more complicated systems.

The problem was next approached by von Zeipel [14]. (See also McGehee [6].) He showed that if the motion of the bodies remains bounded, that is if the diameter or other measure of growth of the system remains bounded, as the motion tends to a singularity, then the singularity must be due to a collision. The converse is also true, a singularity that is not due to a collision must not have motion that remains bounded. This allowed von Zeipel [14] to provide a more intuitive distinction between collision and noncollision singularities than that previously offered by Painlevé [7].

Definition: If a singularity occurs at time $t = \sigma$, and $\lim I < \infty$ as t tends to σ, where I is the moment of inertia, then the singularity is a collision singularity. If σ is a noncollision singularity, however, then $\lim I = \infty$ as t tends to σ.

For a noncollision singularity to occur, it is therefore necessary to have one or more of the bodies escaping to, or coming from, infinity in a finite time. ($\sigma < \infty$, from Section 1.) An ejection or capture is thus identical to a noncollision singularity.

A noncollision singularity seems impossible, since a massive body escaping to infinity in finite time would have to acquire infinite kinetic energy. Xia [15] explained this by saying that since there is no lower bound on the potential energy of the system, there is no *a priori* upper bound on the kinetic energy. The system is closed so no forces act on it, and the total energy, which must thus be constant, is equal to the sum of the kinetic energy and the potential energy. This problem has also only been considered in a Newtonian universe, where relativistic effects do not apply. In this case it would be possible for a body to achieve velocities higher than that of light and up to infinity.

As a consequence of von Zeipel's work, it is also the case that oscillatory motion of the type suggested earlier by Painlevé can never lead to a noncollision singularity, because the motion of the bodies remains bounded.

The analysis of noncollision singularities is relevant because if a system can be found to undergo one, then it would have ejected or captured a body. If it could be shown that a given system could not suffer a noncollision singularity then it could not eject nor capture a body.

3. SINGULARITIES AND COLLISIONS

Singularities and collisions are strongly connected and it is necessary to study them both in relation to this problem. In Section 2 it was stated that for a noncollision singularity to occur $\boldsymbol{r}(t)$ must tend to Δ as t tends to σ, where $\boldsymbol{r}$ is the representative point of the system in phase space. However, it can be seen from the definition of Δ given earlier that configurations with collision singularities also exist on Δ. This suggests in the neighbourhood of any noncollision singularity in real space, there will be at least one collision singularity. The models of noncollision singularities explained in this and subsequent sections show this to be the case.

Donald Saari, who analysed collisions and singularities in a series of papers in the 1970s, is responsible for the recent interest in singularities of Newtonian systems.

Saari [8, 9, 10] showed the improbability of collisions. He proved [10] that the set of initial conditions of the system leading to collisions is a set of measure zero, for all types of collisions and for all n. This work solved Littlewood's 1968 problem [4].

In another paper, Saari [11] rewrote and extended von Zeipel's proof [14] that for a noncollision singularity one or more of the bodies must go to infinity in finite time by providing a minimum rate of how quickly the bodies must go to infinity. An appropriate measure of the growth of the system is the moment of inertia

$$I = \sum_i m_i \mathbf{r}_i^{\ 2} ,$$

where m_i is the mass of the i^{th} body, and $\boldsymbol{r_i}$ the radius vector of the body.

Saari [11] showed that if there is a singularity at time $t = \sigma$ and I is slowly varying as t tends to σ then the singularity is due to a collision. For a noncollision singularity to occur therefore the system must become unbounded at a rapid rate.

As a consequence of this constraint on the growth, Saari showed [11] that for a noncollision singularity bodies in the system must approach each other arbitrarily closely and infinitely often. This is intuitive as a body heading off to infinity by itself is far from the other bodies and so is affected little by them, having near zero acceleration. This means that its velocity is roughly constant. A constant velocity precludes any hope of getting to infinity in a finite time, which is necessary for a noncollision singularity.

Models constructed by later authors take account of this and theories explaining how such counterintuitive motion could arise are discussed in sections 4 and 5.

These results were extended by Saari [12] to examine the 4-body problem. He showed that in this case, the set of initial conditions that lead to a noncollision singularity form a set of measure zero and are thus infinitely unlikely. Combining this with Painlevé's work [7], and Saari's result [10] that collisions are infinitely unlikely, we see that for $n \leq 4$ singularities of any sort are infinitely unlikely and so most orbits exist for all time.

Saari [12] also extended the result in [11] by giving, for the 4-body problem, an analytic expression for how fast the system must grow in order to obtain a noncollision singularity. If a noncollision singularity occurs at $t = \sigma$, then

$$I(t) \geq \ln^{\alpha}(\sigma - t)^{-1}$$

where α is any positive constant, as t tends to σ, and I is the moment of inertia of the system.

He asserted without proof that a similar statement can be found for arbitrary n.

Saari [11] also showed that in the linear n-body problem, where the bodies are constrained to a straight line, noncollision singularities are in a set of measure zero. It is believed that this approach can be extended to all noncollision singularities because the required behaviour of orbits leading to noncollision singularities, as explained in sections 4 and 5, forces the bodies to approach rapidly a lower-dimensional plane in phase space, which is improbable. A demonstration that all noncollision singularities form a set of measure zero remains an open problem today.

4. EVIDENCE FOR NONCOLLISION SINGULARITIES

Mather and McGehee [5] gave supporting evidence to the existence of noncollision singularities. They showed that in the 4-body problem where all the bodies were on a line, it was possible for simple collisions to accumulate in such a way as to eject one body to infinity in finite time. Although this does not help to resolve the question of whether gravitating systems can eject or capture bodies, for which a noncollision singularity is necessary, it suggests that systems ejecting bodies to infinity are not so unimaginable as they might appear.

There are two possibilities for the 4-body case:

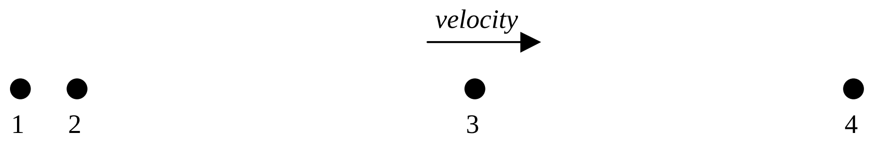


Figure 3 One possibility for collinear motion leading to the ejection of a body to infinity.

In this case a near triple collision occurs between bodies 1, 2 and 3 and body 3 is ejected with an arbitrary velocity toward body 4. Bodies 3 and 4 undergo an elastic collision and body 3 is sent back toward bodies 1 and 2. If it arrives at the correct time for another near triple collision to occur, then the process will repeat and body 3 will be ejected toward body 4 with a higher velocity than it had before. The process will repeat itself until eventually body 3 is emitted to infinity.

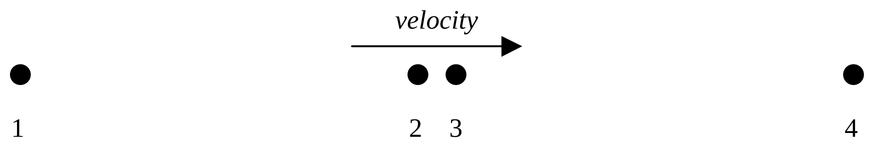


Figure 4 Another possibility for collinear motion leading to the ejection of a body to infinity.

The process shown in Figure 4 is similar to that in Figure 3, except that a pair of bodies travels back and forth, and near triple collisions occur at each end of the system.

In both cases, the speed of the commuting body or bodies increases on each cycle. At each near triple collision, the timing has to be just right in order to ensure that the bodies collide in such a way as to carry on the process. This drastically limits the set of starting configurations that could result in ejection of a body to infinity.

In both cases we can see that this motion fits all the criteria to be a noncollision singularity, except, obviously, that of having no collisions. One or more of the bodies exhibits wildly oscillatory behaviour, the bodies approach each other infinitely often and arbitrarily closely, and the system can be constructed such that the moment of inertia grows fast enough to emit a body to infinity. This gives evidence that although a system fulfilling the stringent criteria for a noncollision singularity is not likely, it may at least be possible.

5. RECENT DEVELOPMENTS

Inspired by Mather and McGehee's construction, Xia [15] used similar techniques to construct a 5-body system where one of the bodies escapes to infinity in finite time without collision. This was the first theoretical construction of a noncollision singularity. A modification of his approach shows that noncollision singularities exist generally in the *n*-body problem for $n \geq 5$.

Xia's 5-body construction is as follows:

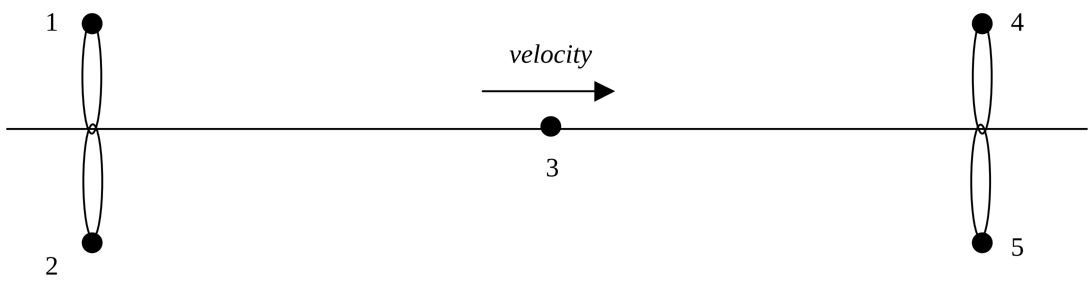


Figure 5 Xia's 5-body construction leading to a noncollision singularity.

Bodies 1 and 2 are coplanar, as are 4 and 5. These bodies' orbits in their respective planes are extremely eccentric ellipses. Bodies 1 and 2 have equal mass, as do bodies 4 and 5, although their mass is not necessarily equal to that of bodies 1 and 2. If bodies 1 and 2 are close to the axis of travel then they exert a large force on body 3. If body 3 arrives at the right time, just missing a triple collision (Figure 6a), then its motion will be braked and it will be pulled back toward bodies 1 and 2, missing them as they separate (Figure 6b), and it will head toward bodies 4 and 5 with an arbitrary velocity (Figure 6c). The same process will then occur with bodies 4 and 5 and body

3 will be re-emitted toward bodies 1 and 2 with a higher velocity. Continued iterations of this process will give body 3 enough kinetic energy to be expelled to infinity in a finite time.

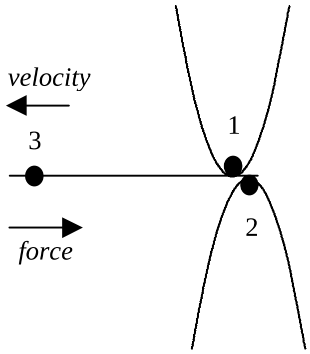


Figure 6a
Bodies 1 and 2 exert a strong force on body 3.

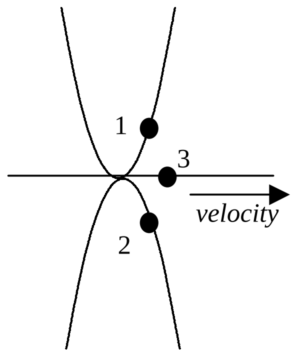


Figure 6b
Body 3 passes between bodies 1 and 2.

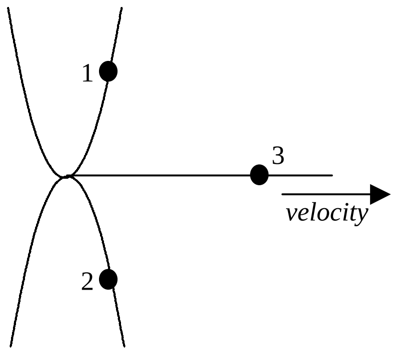


Figure 6c
Body 3 heads toward bodies 4 and 5.

6. CONCLUSIONS

The work in this field may be summed up as follows:

1. If a system of gravitating bodies captures another body, then in almost all cases it will emit a body in the future.
2. A noncollision singularity is equivalent to ejection or capture of a body.
3. In the 3-body case, all singularities are collisions, and no body can be ejected or captured.
4. In the 4-body case, noncollision singularities exist, allowing ejection or capture, but such singularities are infinitely rare.
5. For $n \geq 5$, noncollision singularities exist, although it is not known in general how likely they are.
6. There are special cases, for instance when all the bodies are on a line, for which it is known for general n that noncollision singularities are infinitely rare.

7. FURTHER WORK

Xia has provided [15] an intuitively easy picture of how five bodies may gravitate together to emit one to infinity. Saari [12] has shown that the $n = 4$ case also allows for noncollision singularities, but no similar construction has yet been provided.

In Xia's [15] construction the masses of bodies 1 and 2 are equal, as are the masses of bodies 4 and 5. Although Xia has shown that noncollision singularities exist generally for all $n \geq 5$ it may be possible that there are specific combinations of masses for which noncollision singularities cannot be constructed.

It is suspected that all noncollision singularities form a set of measure zero for general n. A proof would show that it was infinitely unlikely that a system of gravitating bodies would emit or capture another body.

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