How to Parameterize a Two–Dimensional Covariance Matrix

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Abstract

How can this idea be generalized?

The following representation of a positive-definite 2×2 matrix has all the properties we could ask for. All real values of the parameters c, u, and v produce a legal positive-definite matrix; and each such matrix can be produced by exactly one setting of the parameters. There are no discontinuities in the map or its inverse.

$$\mathbf{M}(c, u, v) = \exp(c) \left[\begin{array}{cc} \sqrt{1 + u^2 + v^2} + u & v \\ v & \sqrt{1 + u^2 + v^2} - u \end{array} \right] \tag{1}$$

The matrix part (in $[\cdots]$) has determinant 1.

Another way of expressing the same idea is:

$$\mathbf{M}(c, a, b) = \exp(c) \left[\begin{array}{cc} \exp(a) \cosh b & \sinh b \\ \sinh b & \exp(-a) \cosh b \end{array} \right] \tag{2}$$

The matrix part (in $[\cdots]$) has determinant 1.

A third option is:

$$\mathbf{M}(f_1, f_2, b) = \left[\begin{array}{cc} \exp(f_1) \cosh b & \exp((f_1 + f_2)/2) \sinh b \\ \exp((f_1 + f_2)/2) \sinh b & \exp(f_2) \cosh b \end{array} \right] \tag{3}$$

1 Dimensionally valid version

The rules of dimensions say

1. Only dimensionless expressions may be exponentiated.

2. Added or subtracted quantities must have the same dimensions as each other.

A covariance matrix in general measures the covariance between quantities with different dimensions. The entry M_{ij} has the same dimensions as $\langle x_i x_j \rangle$. Let's introduce n_i, a fixed vector of dimensionally-correct objects defining units in which we could measure x_i. Thus we could write $x_i = z_i n_i$, where z_i is dimensionless.

Then

$$\mathbf{M}(c, a, b) = \exp(c) \begin{bmatrix} n_1^2 \exp(a) \cosh b & n_1 n_2 \sinh b \\ n_1 n_2 \sinh b & n_2^2 \exp(-a) \cosh b \end{bmatrix} \tag{4}$$

and

$$\mathbf{M}(f_1, f_2, b) = \begin{bmatrix} n_1^2 \exp(f_1) \cosh b & n_1 n_2 \exp((f_1 + f_2)/2) \sinh b \\ n_1 n_2 \exp((f_1 + f_2)/2) \sinh b & n_2^2 \exp(f_2) \cosh b \end{bmatrix} \tag{5}$$

Version 1.1

2