Encyclopedia of Sparse Graph Codes

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Abstract

Evaluation of Gallager codes for low error tolerance, short block length and high rate applications.

Sparse graph codes include Gallager codes, Tanner codes, MN codes Repeat–Accumulate codes (RA codes), and turbo codes, all of which have near–Shannon limit performance.

This paper (which is still in preparation) describes empirical properties of a wide selection of these codes, comparing in particular the codes' block error rates with an emphasis on undetected versus detected errors. We explore the dependence of block error rate on block length and other code construction parameters. Histograms of decoding time are also shown.

Draft 1 concentrates on small block lengths.

1 Regular Gallager codes

For each construction and block length I generated three random codes. Their performances are shown in figures 1–4 to give an idea of the variability within one construction.

The program GHG.p was used to make these codes. This program ensures there are no four–cycles in the code's graph. In subsequent figures, one representative (the best) of these three is selected.

1.1 Dependence on transmitted block length N

Figures 5 and 6.

1.2 Dependence on column weight t

Figures 7 and 8.

2 High rate Gallager codes

This section includes a collection of codes with rates $191/273 = 0.7$ and $813/1057 = 0.77$, and a collection of codes with rates greater than 0.89 and block lengths around 2000 and 4000.

2.1 Rates 0.7 and 0.77

2.2 Codes with rates above 0.89

More results on these codes can be found in another publication [3].

2.3 Discussion

In practical systems such as disc drives, people concatenate an outer ECC with an inner run-length-limiting code (RLL code), the latter being small and non-linear. This doesn't seem ideal, since it means that the errors confronting the ECC are rather complex. So, what if we could make an ECC that *is* an RLL code?

An example of such an ECC is (surprise!) a Gallager code. If I make a Gallager code whose top rows go like this:

```
1111100000000000000000000
0000011111000000000000000
0000000000111110000000000
0000000000000001111100000  ....
                        .
                        .
                        .
```

where the row weight, k, is 5 in this example, then the maximum possible run length of 1s in a codeword is 2(k-1) = 8.

If the row weight is even instead of odd, we can get more bang for our buck: if k=6, we can modify every codeword in the code by adding the vector `100000100000100000100000100000100000...` to it. Then all codewords will have a maximum possible run length of 1s equal to $2(k-1)$, *and* a maximum possible run length of 0s equal to $2(k-1)$.

With a minor tweak of the Gallager code, I can reduce $2(k-1)$ above to $2k-3$.

Thus it is easy to make, for example, a good Gallager code with rate $R = 1/2$ that has a maximum run length of 9.

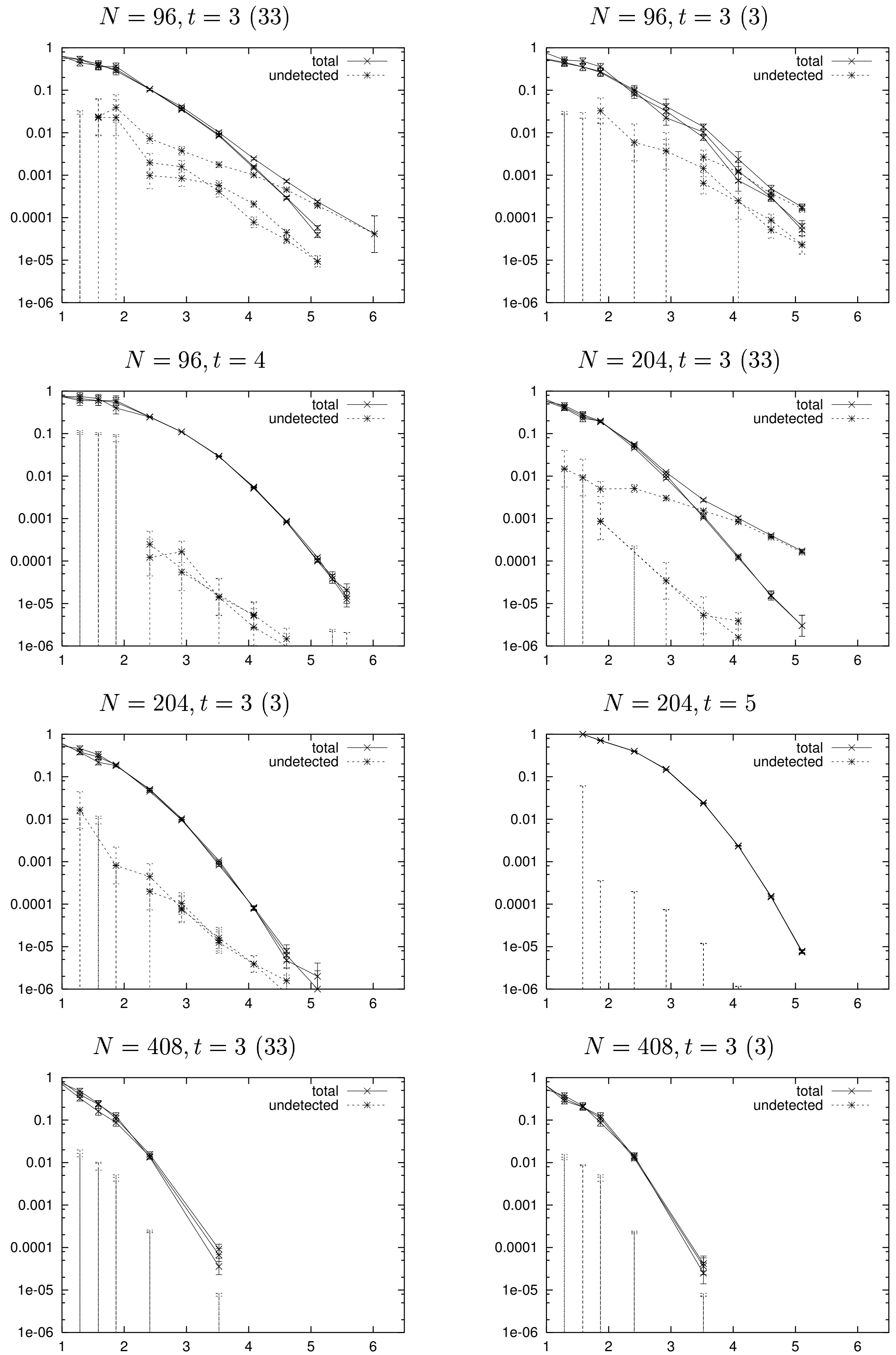

Figure 1: Regular Gallager codes with rate $R = 1/2$. Variability within one construction. Dependence of block error rate on signal to noise ratio, weight per column t and transmitted block length N. Vertical axis: block error rate. Horizontal axis: E_b/N_0 (decibels).

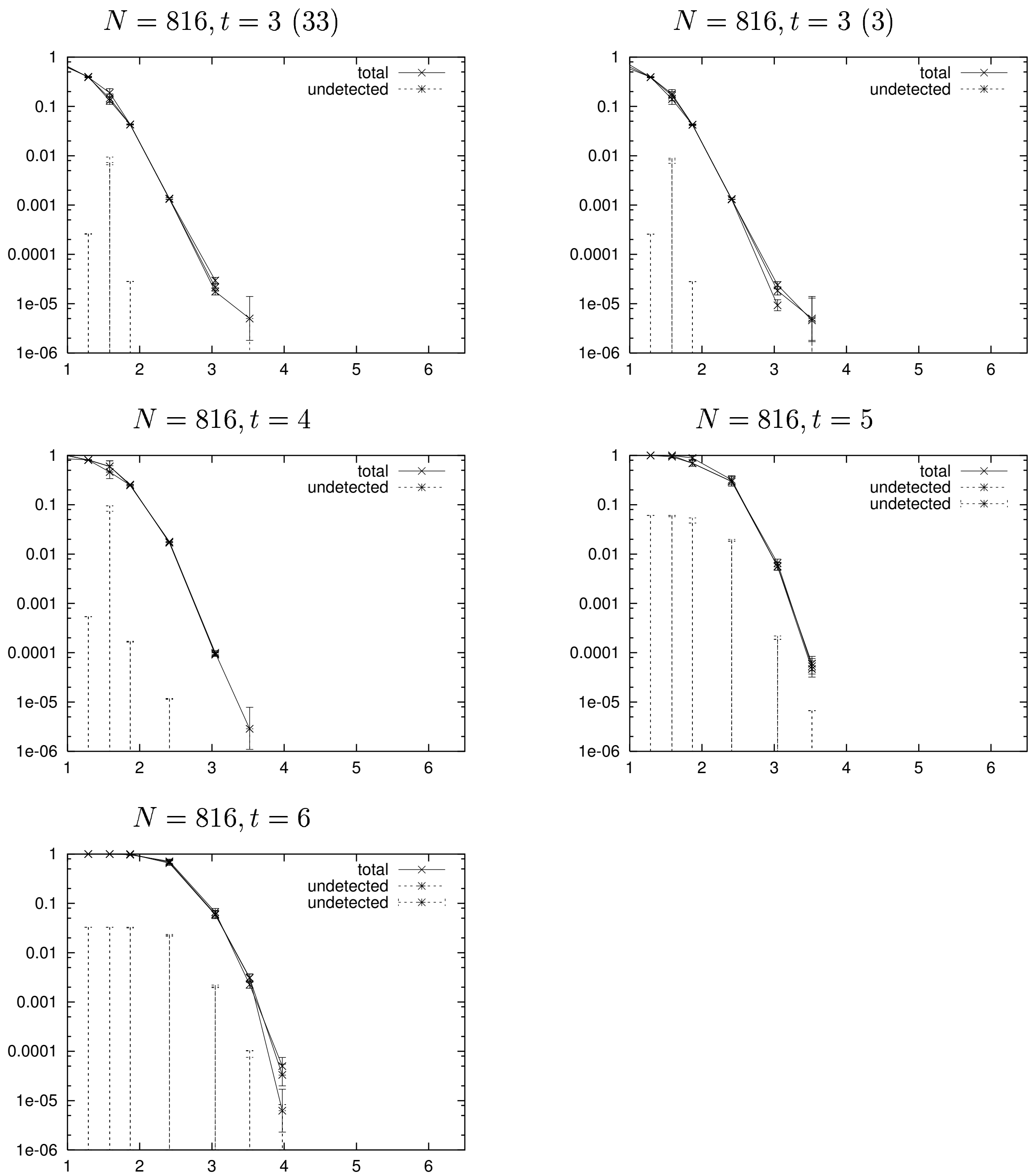

Figure 2: Regular Gallager codes with rate $R = 1/2$. (continued)

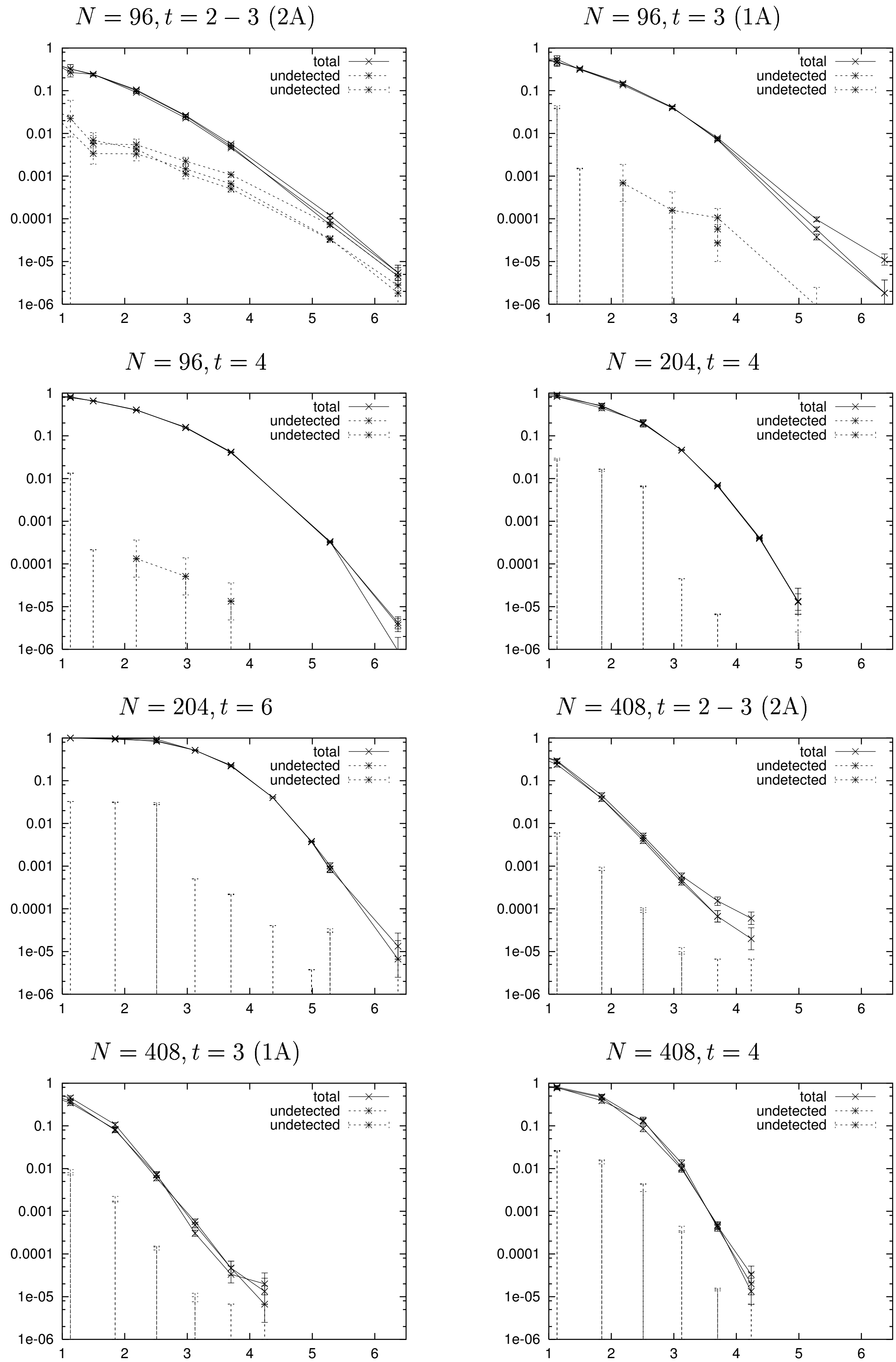

Figure 3: Regular Gallager codes with rate $R = 1/3$. Variability within one construction. Dependence of block error rate on signal to noise ratio, weight per column t and transmitted block length N. Vertical axis: block error rate. Horizontal axis: E_b/N_0 (decibels).

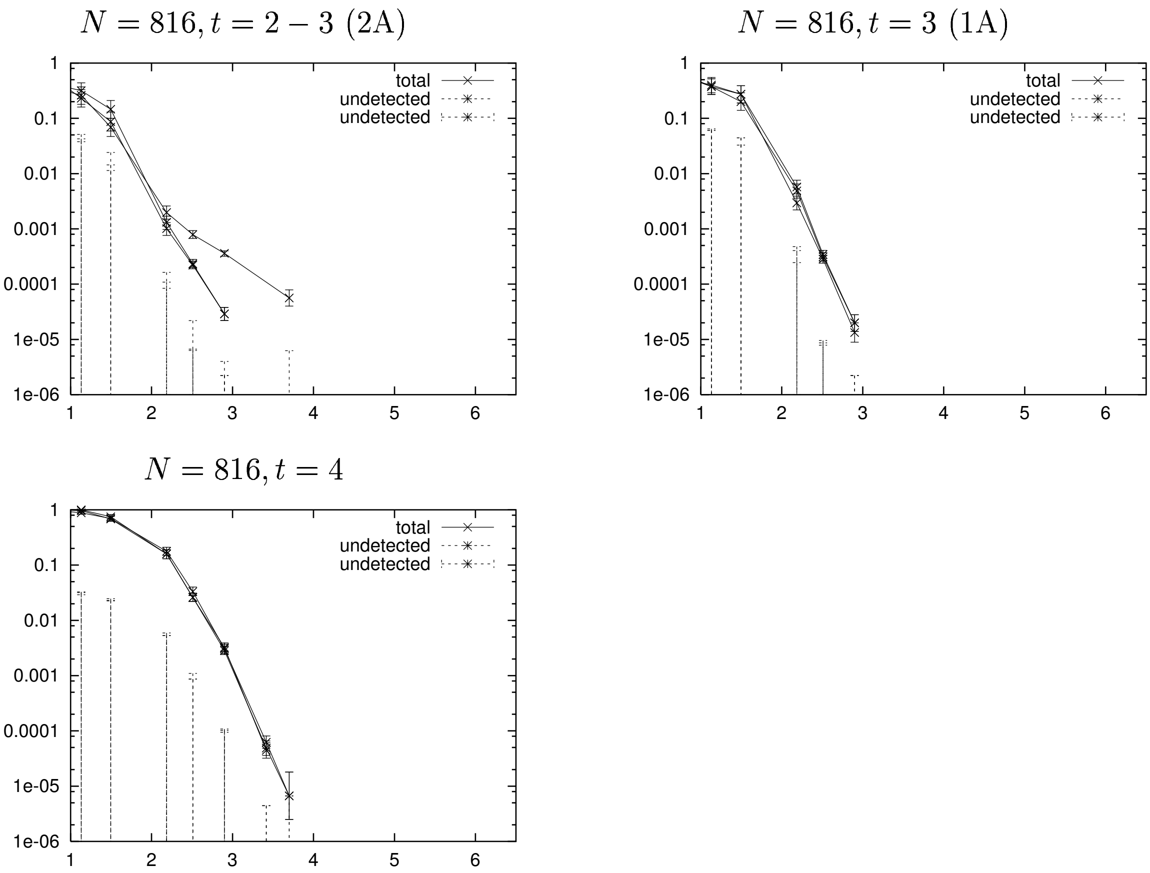

Figure 4: Regular Gallager codes with rate $R = 1/3$. (continued)

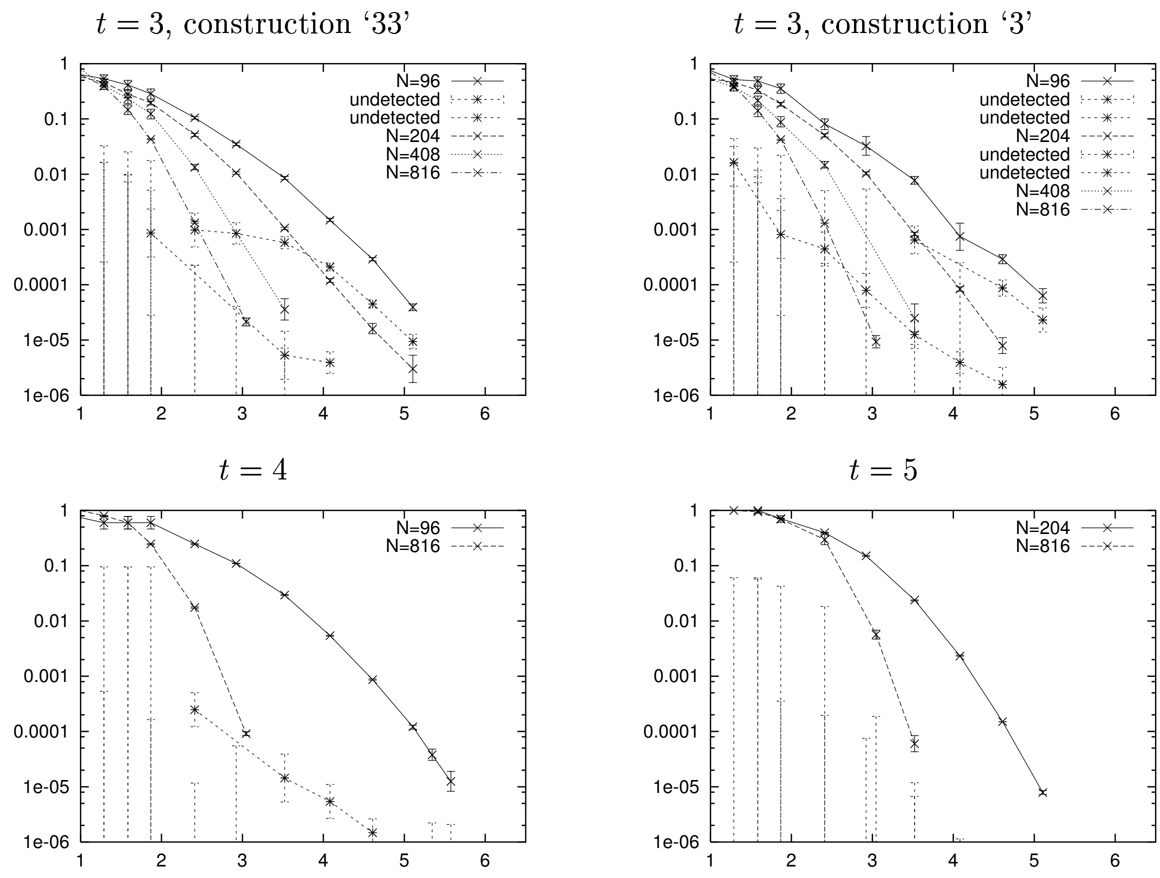

Figure 5: Regular Gallager codes with rate $R = 1/2$. Dependence of block error rate on signal to noise ratio, weight per column t and transmitted block length N. Vertical axis: block error rate. Horizontal axis: E_b/N_0 (decibels).

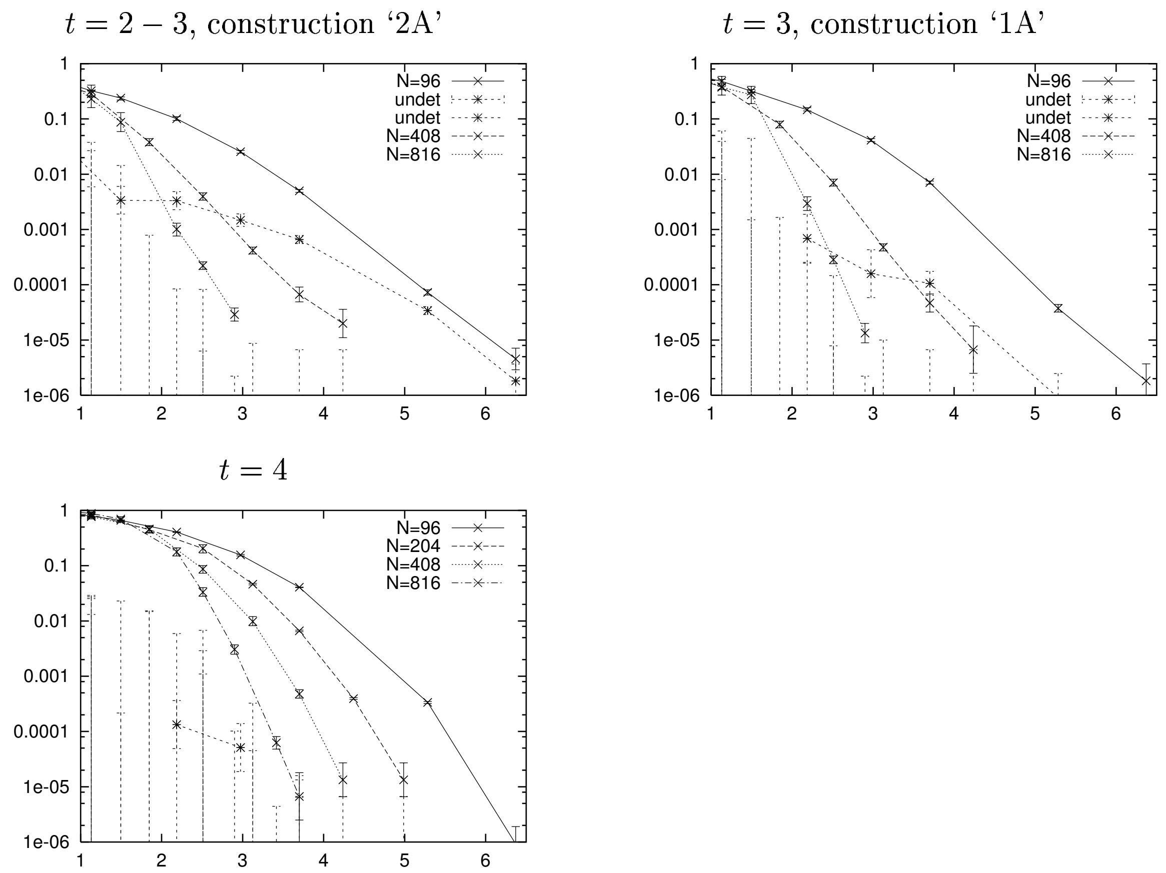

Figure 6: Regular Gallager codes with rate $R = 1/3$. Dependence of block error rate on signal to noise ratio, weight per column t and transmitted block length N. Vertical axis: block error rate. Horizontal axis: E_b/N_0 (decibels).

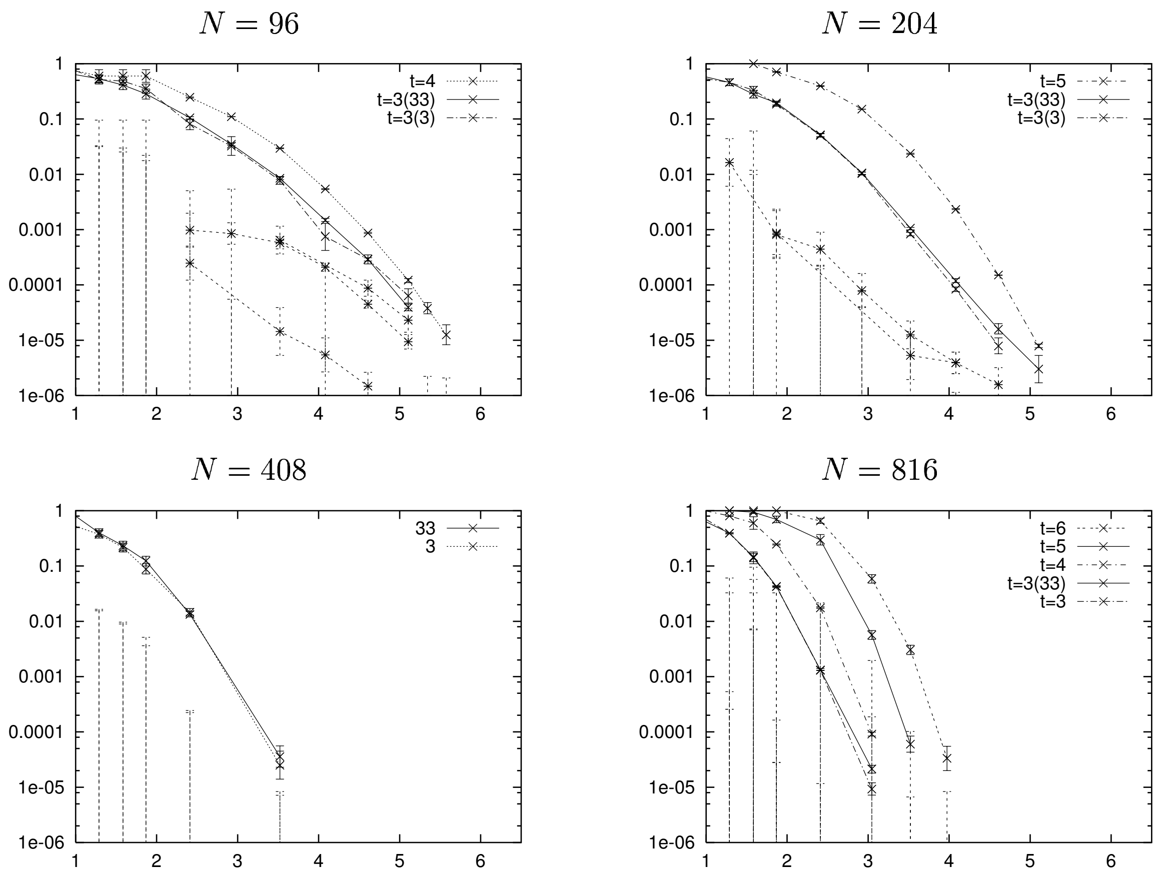

Figure 7: Regular Gallager codes with rate $R = 1/2$. Dependence of block error rate on signal to noise ratio, weight per column t and transmitted block length N. Vertical axis: block error rate. Horizontal axis: E_b/N_0 (decibels).

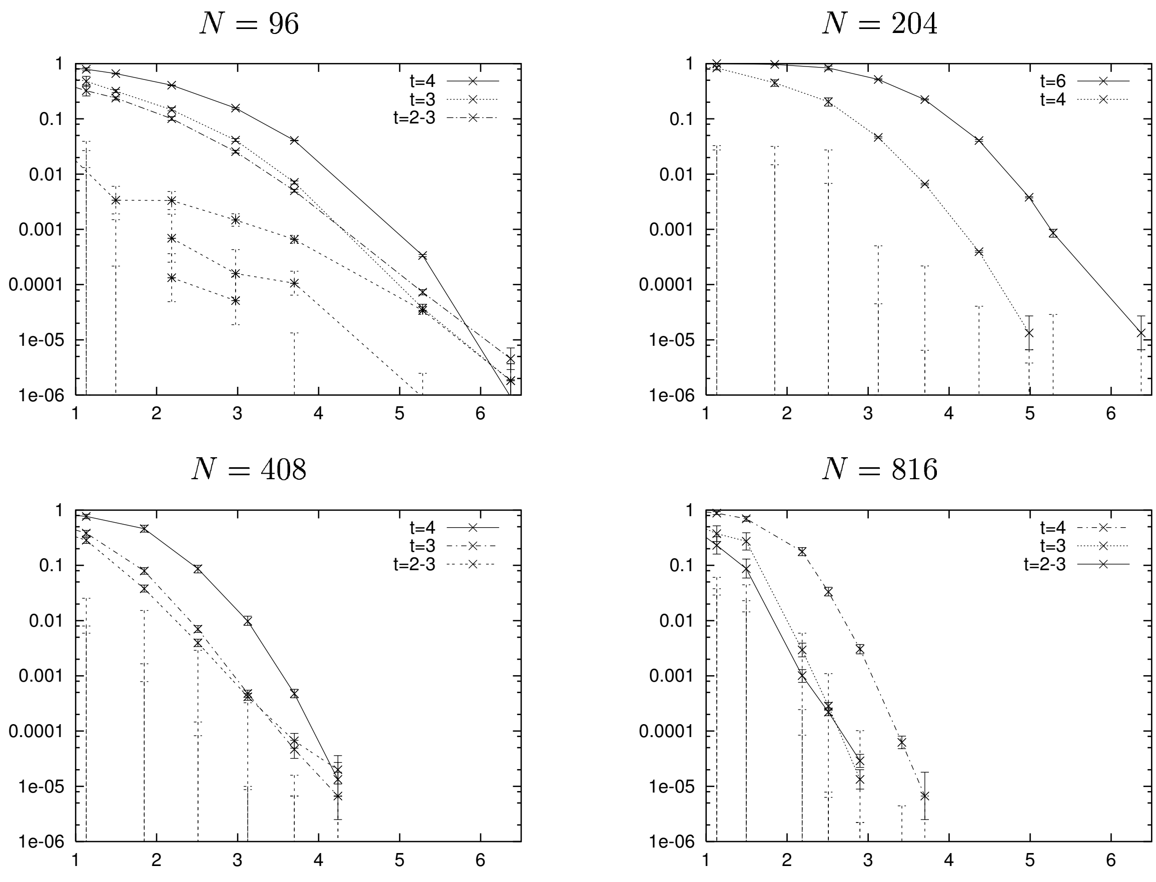

Figure 8: Regular Gallager codes with rate $R = 1/2$. Dependence of block error rate on signal to noise ratio, weight per column t and transmitted block length N. Vertical axis: block error rate. Horizontal axis: E_b/N_0 (decibels).

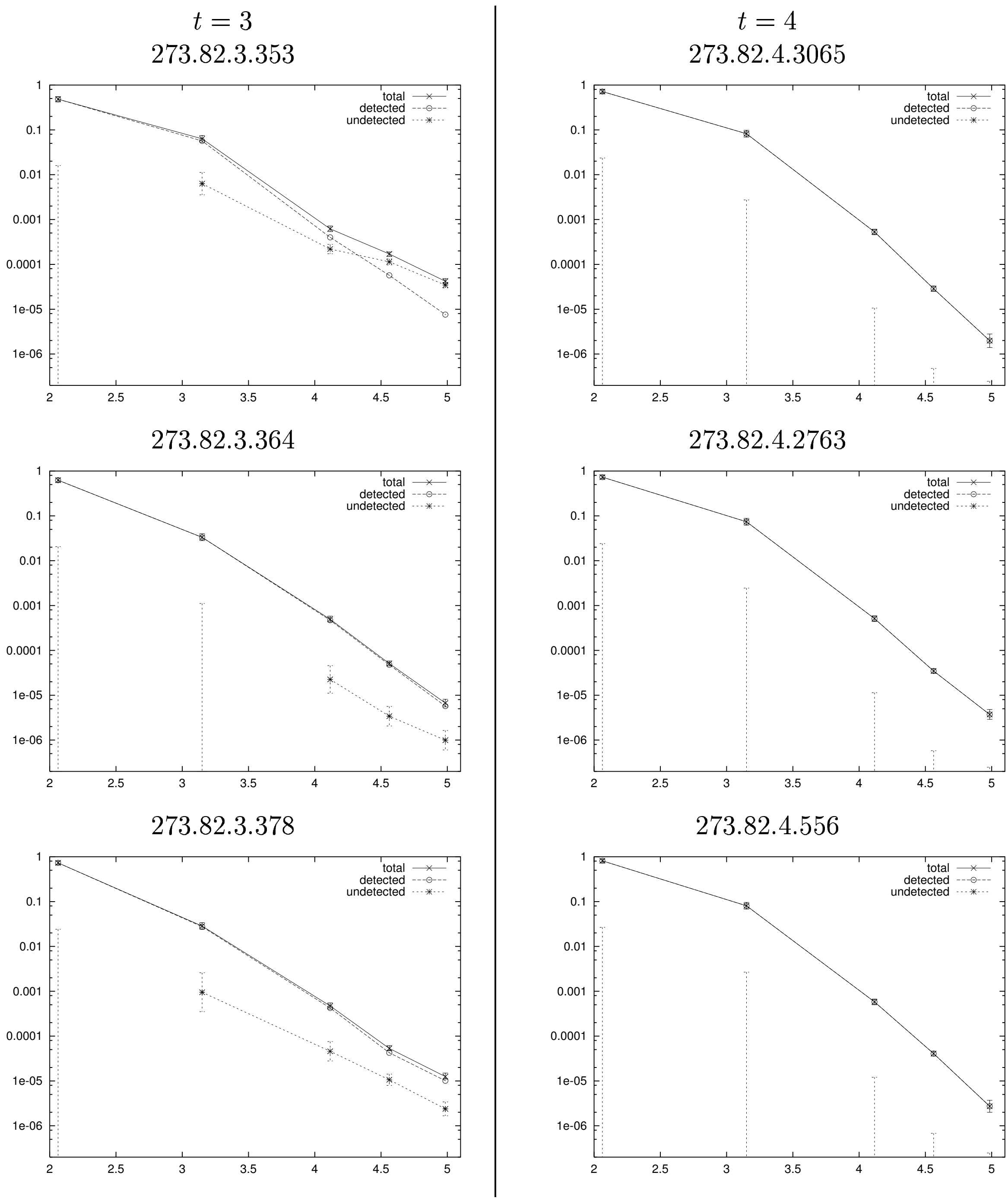

Figure 9: Regular Gallager codes with rates $R = K/N = 191/273 = 0.7$. Dependence of block error rate on signal to noise ratio. Weight per column is $t = 3$ or $t = 4$. Vertical axis: block error rate. Horizontal axis: E_b/N_0 (decibels).

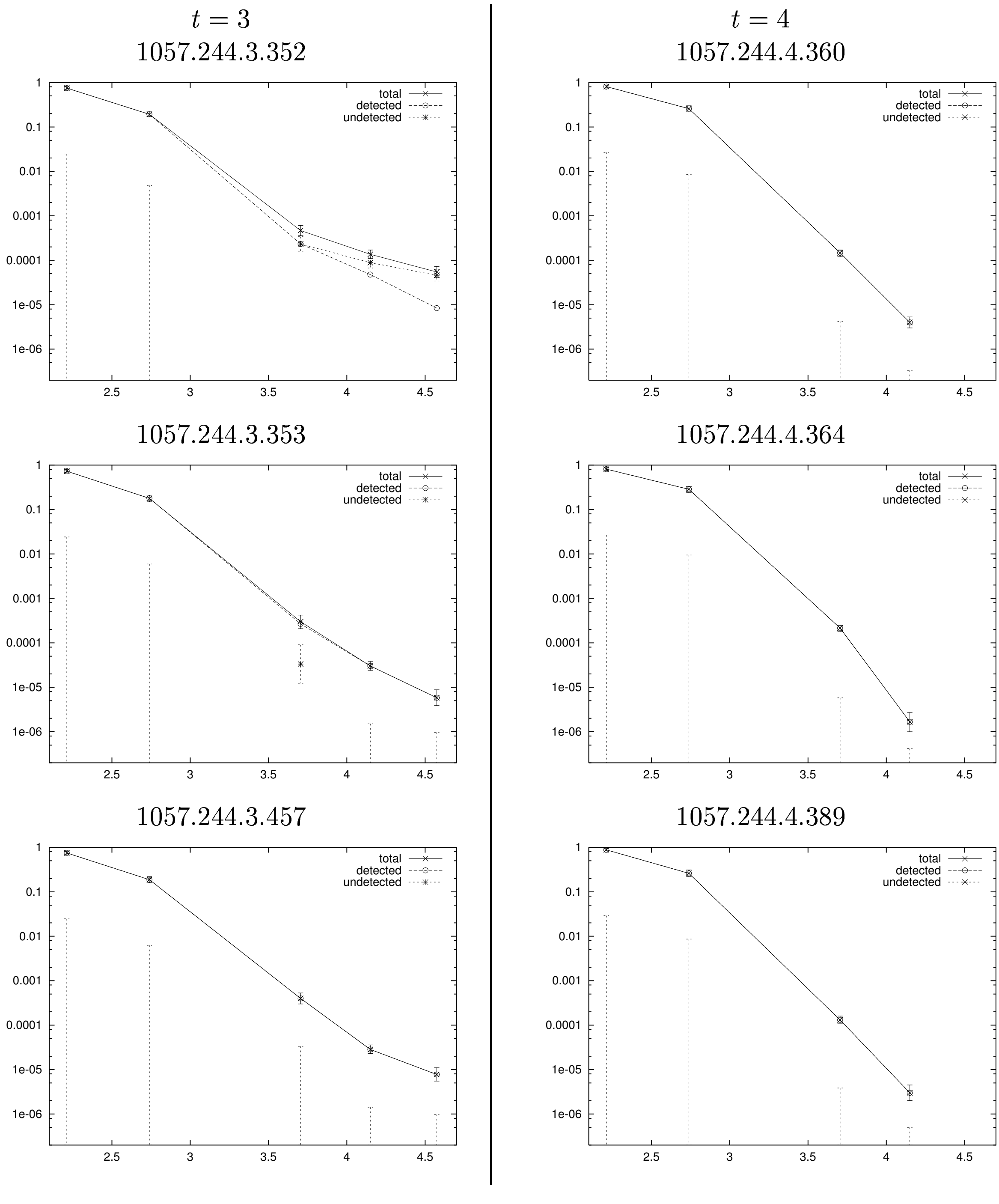

Figure 10: Regular Gallager codes with rate $R = 0.77$, $K = 813$, $N = 1057$. Dependence of block error rate on signal to noise ratio. Weight per column is $t = 3$ or $t = 4$. Vertical axis: block error rate. Horizontal axis: E_b/N_0 (decibels).

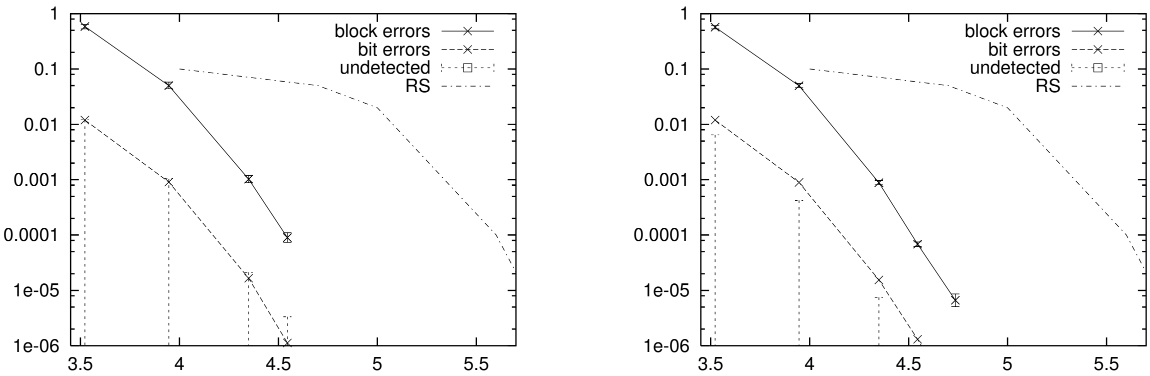

Figure 11: Regular Gallager codes with rate $R = 8/9$. Dependence of block error rate on signal to noise ratio. Weight per column $t = 4$ and transmitted block length $N = 1998$. Vertical axis: block error rate. Horizontal axis: E_b/N_0 (decibels).

3 Repeat–accumulate codes

3.1 Dependence on transmitted block length N

3.1.1 Method

For each block length, between one and five random codes were constructed to assess the variability of the performance within the code family for fixed block length (lower panels of figure 12). The best code for each block length is shown in the upper panel of figure 12.

The histogram of decoding times τ for Gallager codes have been found to follow a power law for large τ [2]. Figures 14(b–c) show power laws fitted by hand to the histograms of figures 13(b–c). Histogram (b) is well fitted by $1/\tau^6$ and (c) is well fitted by $1/\tau^9$.

Figure 13(d) shows the block error probability versus signal to noise ratio for the same RA code of source length $K = 10000$ and transmitted block length $N = 30000$. Most errors in this experiment were detected errors, but at high signal–to–noise ratios some undetected errors occur (one so far, corresponding to a codeword of weight 18, found after 8000 block transmissions).

In my experience RA codes with smaller block lengths make more undetected errors. I'm going to investigate further these undetected errors (which sometimes correspond to codewords and sometimes to *near codewords*).

Discussion

The Divsalar, Jin and McEliece paper [1] contains graphs for up to 30 iterations. Histogram (a) above shows that roughly 50% of the decodings in my simulation at 0.65dB took more than 30 iterations, with a small but substantial number taking more than 60 iterations. So if you ran the old–fashioned decoder for a whopping twice as long, you could cut the block error probability by a factor of ten or so. However, using the stop–when–it's–done decoder, you can use roughly the same amount of computer time as the original simulation and get the error probability *even lower*.

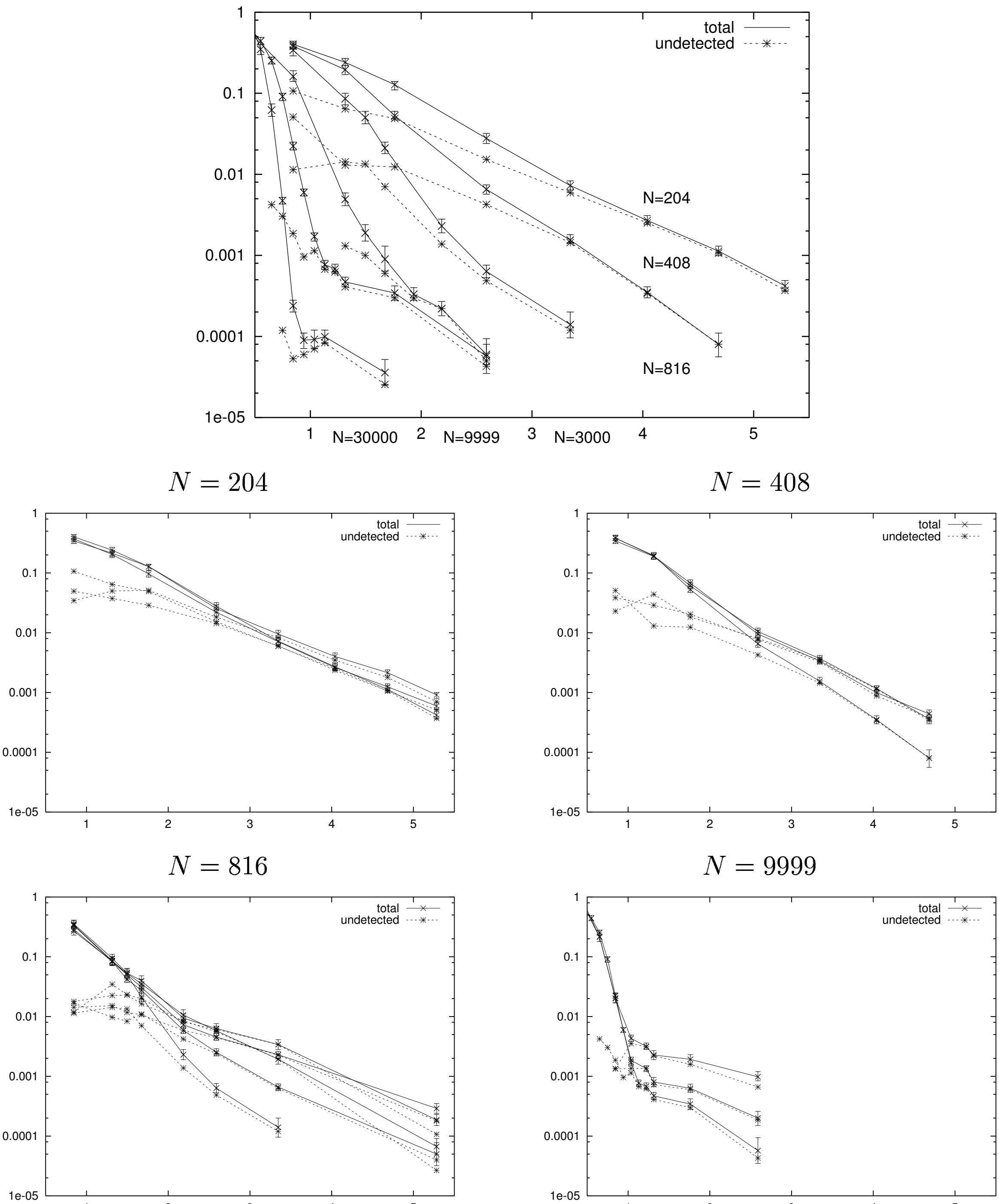

Figure 12: Repeat–accumulate codes with rate $R = 1/3$. Dependence of block error rate on signal to noise ratio and transmitted block length N. The source block length K is equal to one third of N. Vertical axis: block error rate. Horizontal axis: E_b/N_0 (decibels).

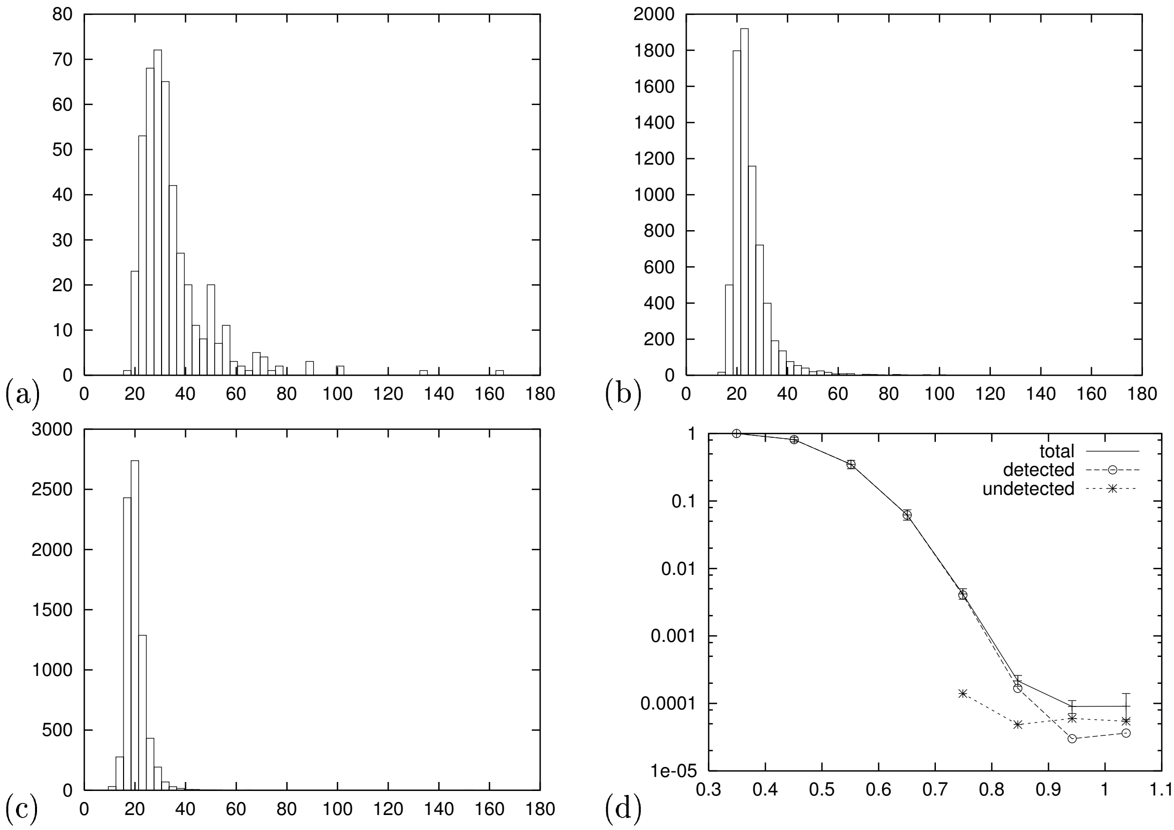

Figure 13: Histograms of number of iterations to find a valid decoding. The RA code has source block length $K = 10000$ and transmitted block length $N = 30000$. (a) Channel signal to noise ratio $x/\sigma = 0.88$, $E_b/N_0 = 0.651$ dB. (b) $x/\sigma = 0.89$, $E_b/N_0 = 0.749$ dB. (c) $x/\sigma = 0.90$, $E_b/N_0 = 0.846$ dB. (d) Block error probability versus signal to noise ratio for the RA code. Most errors in this experiment were detected errors, but not quite all.

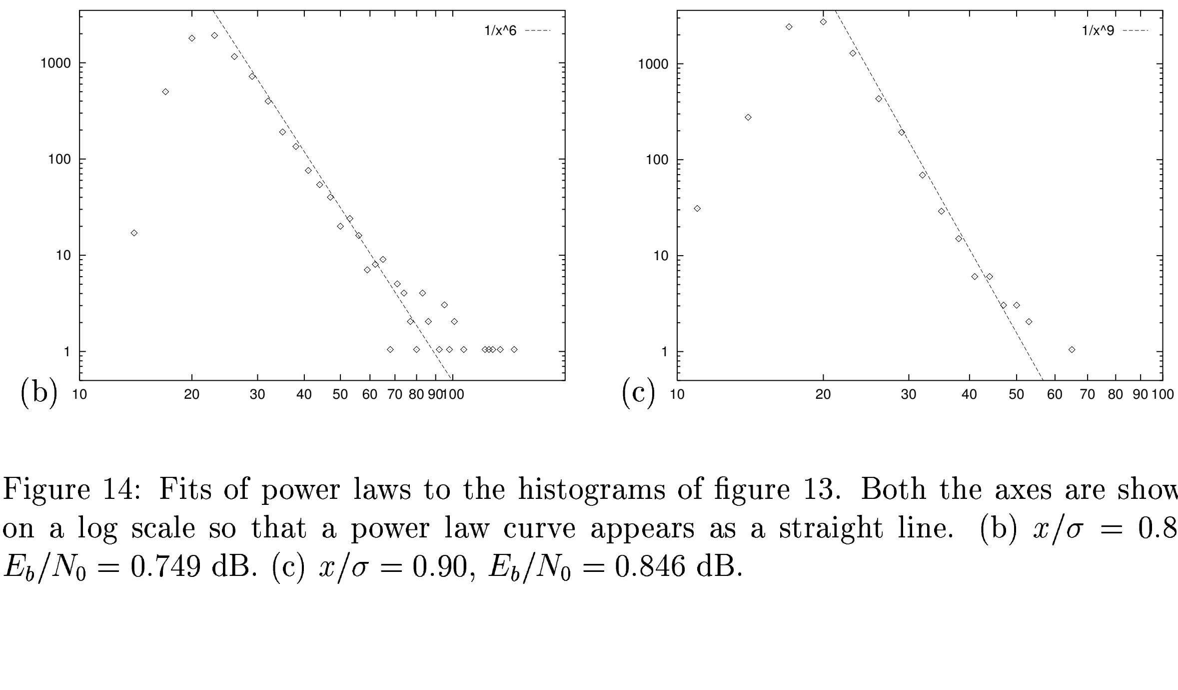

Figure 14: Fits of power laws to the histograms of figure 13. Both the axes are shown on a log scale so that a power law curve appears as a straight line. (b) $x/\sigma = 0.89$, $E_b/N_0 = 0.749$ dB. (c) $x/\sigma = 0.90$, $E_b/N_0 = 0.846$ dB.

Nothing is lost, because (if you log the stopping time in a file) you can always recover the old–fashioned graphs if anyone still wants them.

The stop–when–it's–done method

'OK', I hear you say, 'but how do you detect a valid decoder state for an RA code? — Surely any hypothesis about the K source bits is a valid hypothesis?'

The method I used is as follows.

1. While running the sum–product algorithm up and down the accumulator trellis, note the most probable state at each time. (You can get this by multiplying the forward signals by the backward signals.) This state sequence — if you unaccumulate it — defines a sequence of guesses for the source bits, with each source bit being guessed q times. (q is the number of repetitions in the RA code.)

2. When reading out the likelihoods from the trellis, and combining the q likelihoods for each of the source bits, compute the most probable state of each source bit. This is the state which maximizes the product of the likelihoods.

3. If *all* the guesses in (1) agree with the most probable state found in (2) then the decoder has reached a valid state. Halt.

The cost of these extra operations is small compared to the cost of decoding.

RA code construction method

In case you want to know how I made the RA code, I simply permuted the source block of length K twice using two randomly chosen $K \times K$ permutations in order to create the scrambled repeated signal. This isn't the same as repeating three times and scrambling the whole block of size N, but I expect it makes little difference when the block length is large.

Results for smaller block lengths

Figure 15 shows results for various block lengths. It also shows results for three codes all having block length $N = 9999$ to assess the variability from code to code in a single ensemble.

3.2 Acknowledgements

This work was supported by the Gatsby Foundation. DJCM thanks the researchers at the Sloane Center, Department of Physiology, University of California at San Francisco, for their generous hospitality, and Lucent and Seagate for helpful discussions.

References

[1] D. Divsalar, H. Jin, and R. J. McEliece. Coding theorems for 'turbo–like' codes. 1998.

[2] D. J. C. MacKay. Deccoding times of irregular Gallager codes. Unpublished, 1998.

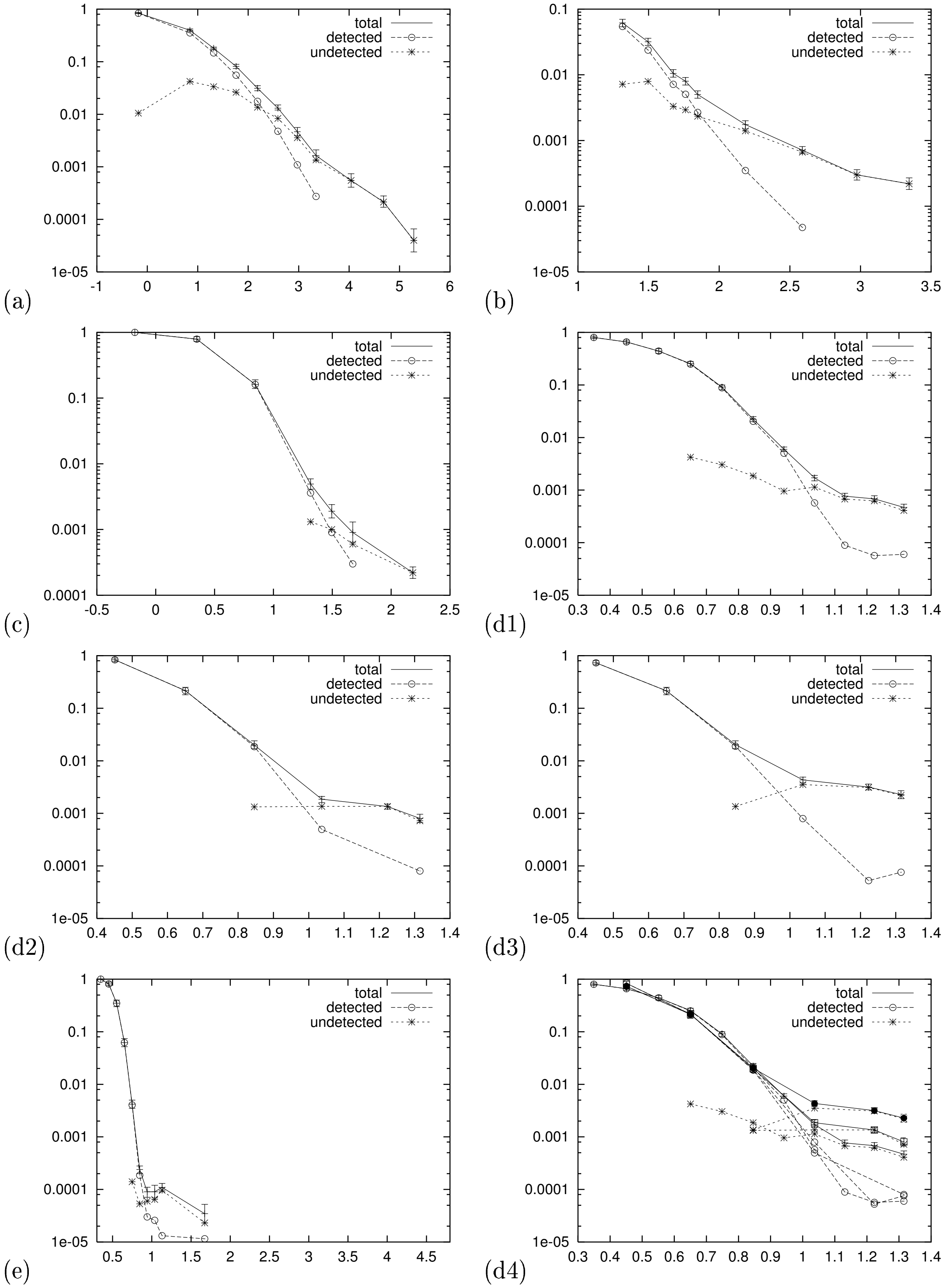

Figure 15: Block error probability versus signal to noise ratio for RA codes with rate 1/3. (a) Source block length $K = 100$, transmitted block length $N = 300$. (b) $K = 333$, transmitted block length $N = 999$. (c) $K = 1000$, transmitted block length $N = 3000$. (d1–3) $K = 3333$, transmitted block length $N = 9999$. (Three codes shown to illustrate the variability from code to code sharing identical parameters.) (e) $K = 10000$, $N = 30000$ (already shown in figure 13(d)). (d4) Superposition of (d1–3).

[3] D. J. C. MacKay. Evaluation of gallager codes for short block length and high rate applications. Available from `www.keck.ucsf.edu/~mackay/`, 1998.