Gallager Codes for High Rate Applications

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Abstract

Gallager codes with large block length and low rate (*e.g.*, $N \simeq 10,000$–$40,000$, $R \simeq 0.25$–0.5) have been shown to have record-breaking performance for low signal-to-noise applications. Gallager codes with high rates ($R \simeq 0.8$–0.94) are also excellent, outperforming comparable BCH and Reed-Solomon codes, even at short block-length ($N \simeq 2,000$–$4,000$) (MacKay and Davey, 2000).

This paper looks in more detail at high-rate Gallager codes, addressing the following issues. First, we recap the performance improvement obtained by switching from a Reed-Solomon code to a comparable Gallager code. Second, we investigate the benefit of increasing the blocklength of the Gallager code, and investigate the variability in performance among randomly-created codes. Third, we show the benefit of reducing the rate of the code, *i.e.*, increasing its redundancy. Fourth, we describe modifications to the code that are appropriate if there are many parallel channels that contaminate each other with crosstalk, as might be the case in an optical channel and quantify the benefits of these changes.

1 Introduction

A regular Gallager code (Gallager, 1962) has a parity check matrix with uniform column weight j and uniform row weight k, both of which are very small compared to the blocklength. If the code has transmitted blocklength N and rate R then the parity check matrix $\mathbf{H}$ has N columns and M rows, where $M \geq N(1-R)$. [Normally parity check matrices have $M = N(1-R)$, but the matrices we construct may have a few redundant rows so that their rate could be a little higher than $1 - M/N$.]

[The 'overhead' of a code is given by $(1 - R)/R$.]

Rate	Overhead
0.5	100%
0.8	25%
0.85	18%
0.94	6.4%

In (MacKay and Davey, 2000), we explored whether Gallager codes were useful for high rates ($R > 2/3$) and small block lengths ($N < 5000$), and showed that they could outperform Reed-Solomon codes. The present paper continues from that work.

2 Comparison with Reed-Solomon codes and BCH codes

We recap a result from (MacKay and Davey, 2000), comparing the performance of a rate-239/255 Reed-Solomon code with a Gallager code. The blocklength of the Reed-Solomon code is $N = 2040$. The Gallager code has $N = 4376$, $M_{\text{apparent}} = 282$, $j = 4$, and $M_{\text{true}} = 281$. (Rate 0.936, c.f. $239/255 = 0.937$.) It was created using David MacKay's `code5` program to generate a parity-check matrix with almost no 4-cycles, then pruning appropriate columns from the parity-check matrix to remove all 4-cycles. Empirical results are based on 18,650,000 simulated block decodings. Error bars are shown. No undetected errors occurred.

The comparison is rather unfair on the Reed-Solomon code for the following reasons.

1. Our comparison is on a Gaussian-additive-noise channel, but the standard Reed-Solomon code's decoder cannot make use of soft channel outputs. The Gallager code's decoder makes excellent use of the soft outputs.

2. The Reed-Solomon code is designed to handle bursty noise. The channel simulated here has no bursts.

[Even if we remove these unfairnesses, Gallager codes can still beat Reed-Solomon codes, as we showed in (MacKay and Davey, 2000), where we simulated Gallager codes over GF(16) with a simple bursty channel model.]

Figure 1 shows the block error probabilities of the two codes, and of two BCH codes with similar rate.

Gallager codes beat Reed-Solomon codes and BCH codes because they not only have good distance properties, they also have a soft decoding algorithm that can (almost always) correct errors at noise levels *greater* than those that would be tolerated by any bounded-distance decoder. Reed-Solomon codes and BCH codes only have bounded-distance decoders, for practical purposes.

3 Blocklength

We now investigate the effect of increasing blocklength. Figure 2 shows the block error probabilities of three additional codes, all of essentially identical rate, but having blocklength $N = 10,000$. A win of about 0.5 dB can be seen at a block error rate of 10^{-6}. At smaller block error rates, the gap appears to widen.

The three codes were created at random using `code5` to make a matrix with $N = 10,002$ columns, following by pruning of one or two columns. The right hand panel of figure 2 illustrates the variability of performance

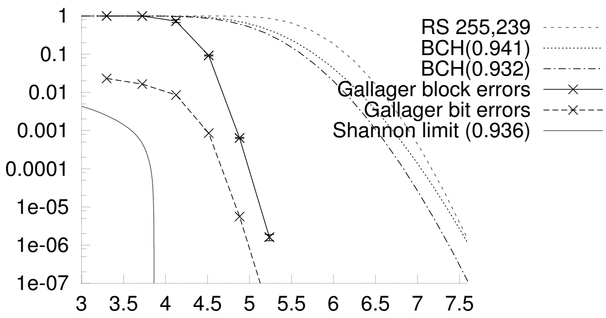

Figure 1: Block error rates of a Gallager code, a Reed-Solomon code, and two BCH codes. [Bit error rates will not be shown in the remaining figures. They are always about a factor of 100 lower than the block error rate.]

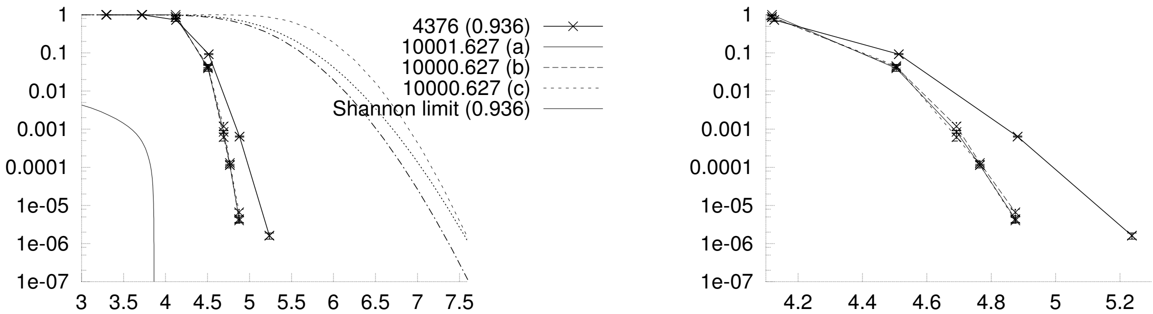

Figure 2: Block error rates as a function of blocklength. The right hand figure shows detail from the left. The codes have parameters (a) $j = 4$, $K = 9375$, $N = 10001$; (b,c) $j = 4$, $K = 9374$, $N = 10000$.

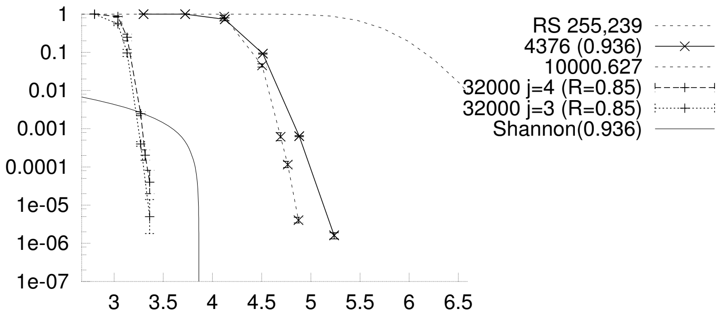

Figure 3: Block error rates as a function of rate. This figure has a different horizontal axis from the preceding figures.
The codes with rate 0.85 have a performance that is beyond the Shannon limit for codes of rate 0.936.

among randomly created codes – much less than 0.1dB, for codes of this size.

4 Choice of Rate

A well-known prediction of Shannon theory is that one can achieve better communication over a Gaussian channel by transmitting at a *lower* code rate and with consequently worse signal-to-noise ratio. I believe this recommendation is generally avoided in the communications world either for practical reasons such as bandwidth limitations, or because of tradition. When we are considering changing the coding paradigm, I think it is important to consider changing the code rate. If one of the reasons for choosing a rate of 0.94 was because that is a rate that is convenient for Reed-Solomon codes, then this constraint on rate is one that we can now ignore!

To illustrate the potential benefit of a reduction in the code rate, figure 3 shows the performance of two randomly-created Gallager codes with rate 0.85. (This is still quite a high rate, by my standards!) The code parameters are $N = 32,000$ and $M \simeq 4,800$, with column weights $j = 3$ and $j = 4$ respectively.

Decreasing the rate from 0.936 to 0.85 delivers a 1.3dB gain on the Gaussian channel.

These gains continue as one reduces the rate to 0.5 or 0.25.

The parity check matrix of this code, `4376.282.4.9598`, can be found in the online archive (MacKay, 1999).

Decoding times.

Figure 4 shows the cumulative distribution of decoding times for the code `s2.94.594` at three noise levels. The decoding usually halts in fewer than ten iterations. Under good conditions three iterations usually suffice.

The number of arithmetical operations per iteration is about four times the number of 1s in the parity check matrix. That makes 16 operations per iteration per transmitted bit, or 32000 operations per iteration if $N = 2000$.

5 Codes for parallel channels with crosstalk

Finally, we describe an evaluation of the benefits of incorporating knowledge about crosstalk into the code design.

As part of Edward Ratzer's PhD, various channels with crosstalk have been studied. The studied model that most closely resembles a wave-division multiplexed (WDM) fibre channel is the parallel Z channel; a sub-channel is shown in figure 5. The noise level in each sub-channel is

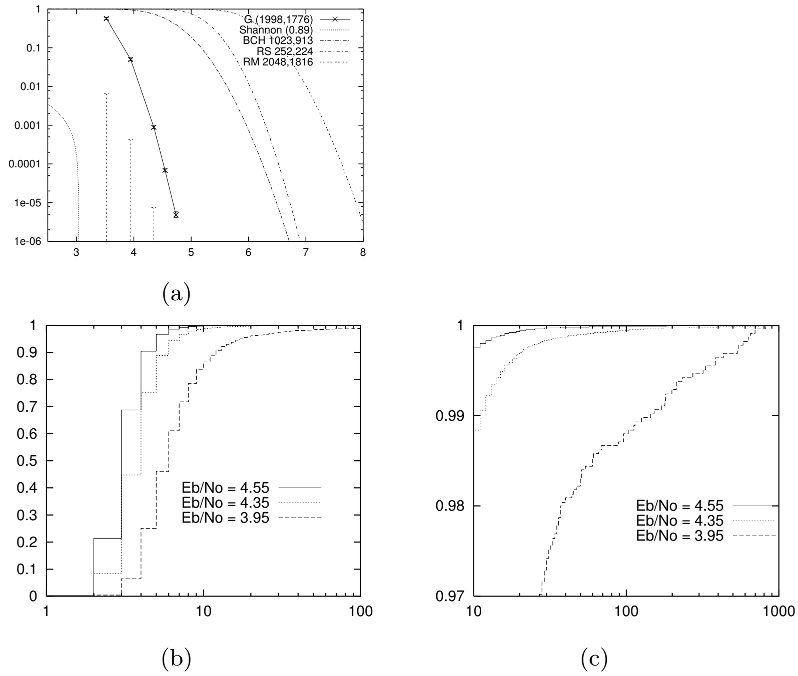

Figure 4: Regular Gallager code with rate $R = 8/9$ and $N = 1998$. (a) Dependence of block error rate on signal to noise ratio. Weight per column $t = 4$ and transmitted blocklength $N = 1998$. Vertical axis: block error rate. Horizontal axis: E_b/N_0 (decibels). Also shown are performance curves for Reed-Solomon, Reed-Muller and BCH codes with similar rate. (b) Decoding times, cumulative distribution. Horizontal axis: number of iterations of the sum-product algorithm. (c) Detail from (b).

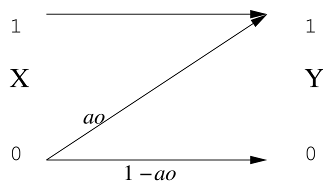

Figure 5: Model of parallel channels with crosstalk. The noise level of each sub-channel is proportional to the number of ones transmitted, o, on the other sub-channels. The constant of proportionality is called a.

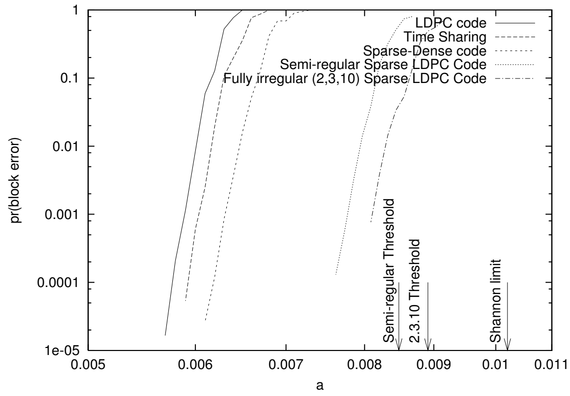

Figure 6: Comparison of five coding strategies for the parallel channels with crosstalk. The codes are all rate 0.3 and blocksize 10000 and the number of parallel sub-channels is 128. The horizontal axis is the noise parameter a defined in figure 5.

proportional to the number of 1s, o, being transmitted on the other channels at the time. If each channel is treated independentally and identically, capacity calculations suggest that a coding solution in which 1s and 0s are not equally likely to be transmitted will lead to better performance.

Simulations were carried out with standard LDPC codes compared to codes with sparse bits (*i.e.*, bits where 1s are less common than 0s) and the results are shown in figure 6. A range of schemes that improve the performance are shown. First, a simple time-sharing scheme, whereby some of the transmitters sit idle for short time shows a small improvement. A further improvement, called a Sparse-Dense code, involves fixing the user data bits to be sparse, but still using a traditional systematic LDPC code. Much larger gains are delivered by a scheme that modifies the LDPC code so that all the bits transmitted are sparse, a Sparse LDPC Code.

Sparse-Dense codes show a similar level of complexity to LDPC codes. For the further gain to Sparse LDPC Codes a higher level of decoding complexity is involved.

These simulations are based on an ad hoc channel model of crosstalk – results of tests on real fibre systems are needed to be able to say what sort of gain will be seen on real channels.

6 Conclusion

We have illustrated gains of several decibels as follows:

Comparison	Gain / dB
Switching from Reed-Solomon to Gallager codes	2.0
Increasing blocklength from 4,000 to 10,000	0.5
Reducing code rate (*i.e.*, increasing redundancy)	1.3
Sparsification, assuming crosstalk	1.2

Further ideas that may be worth investigating include:

1. Making use of the Gallager code's ability to *detect* its decoding failures to invoke a **retransmission** of each lost block. (This approach relies on the fact that the Gallager code can detect essentially *all* its errors, a property not shared by all codes.) Retransmissions offer a cheap and simple way to get the error probability down from 10^{-5} to 10^{-20} with very small increase in overhead. This approach also allows more corners to be cut in the decoding of the Gallager code, since a small increase in the number of undecoded blocks would no longer matter. This suggestion's only cost is a decoding latency, and obviously the requirement for a back-channel over which retransmissions are requested.

2. If retransmissions are not possible, an alternative approach is to concatenate the Gallager code with an outer erasure-correcting code such as a Digital Fountain (Byers *et al.*, 1998). The digital fountain adds negligible computational cost, a small increase in overhead, and a significant decoding latency (of order 1,000–10,000 blocks).

3. Making the Gallager code irregular, and/or introducing state variables into its graph. At long block lengths these modifications can give improvements of 1 or 2 dB (Richardson *et al.*, 2001; MacKay *et al.*, 1999; Davey and MacKay, 1998).

4. Using low-precision decoders. As an example, 'Gallager's decoding algorithm B' is a decoder that passes single bits in place of real numbers. The loss in performance for this decoder is about 2 dB. Intermediate precision algorithms (using say three or five levels for all messages) suffer a much smaller loss (Richardson and Urbanke, 2001).

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