```
1st                AAB ABA ABB ABC           BBC BCA BCB BCC
2nd weighings put  AAB CAA CAB CAC in pan A, ABA ABB ABC BBC in pan B.
3rd                ABA BCA CAA CCA           AAB ABB BCB CAB
```

Now in a given weighing, a pan will either end up in the

- Canonical position (C) that it assumes when the pans are balanced, or
- Above that position (A), or
- Below it (B),

so the three weighings determine for each pan a sequence of three of these letters.

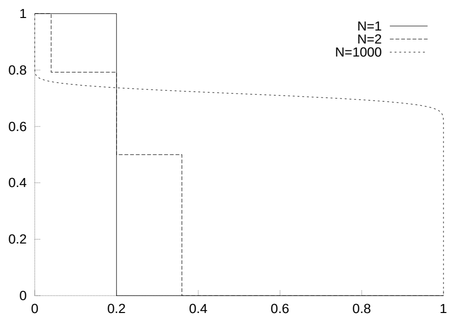
If both sequences are CCC, then there's no odd ball. Otherwise, for *just one* of the two pans, the sequence is among the 12 above, and names the odd ball, whose weight is Above or Below the proper one according as the pan is A or B.

(b) In W weighings the odd ball can be identified from among

$$N = (3^W - 3)/2 \tag{4.49}$$

balls in the same way, by labelling them with all the non-constant sequences of W letters from A, B, C whose first change is A-to-B or B-to-C or C-to-A, and at the wth weighing putting those whose wth letter is A in pan A and those whose wth letter is B in pan B.

Solution to exercise 4.15 (p.85). The curves $\frac{1}{N}H_\delta(X^N)$ as a function of δ for $N = 1, 2$ and 1000 are shown in figure 4.14. Note that $H_2(0.2) = 0.72$ bits.



$N = 1$		
δ	$\frac{1}{N}H_\delta(\mathbf{X})$	$2^{H_\delta(\mathbf{X})}$
0–0.2	1	2
0.2–1	0	1

$N = 2$		
δ	$\frac{1}{N}H_\delta(\mathbf{X})$	$2^{H_\delta(\mathbf{X})}$
0–0.04	1	4
0.04–0.2	0.79	3
0.2–0.36	0.5	2
0.36–1	0	1

Figure 4.14. $\frac{1}{N}H_\delta(\mathbf{X})$ (vertical axis) against δ (horizontal), for $N = 1, 2, 100$ binary variables with $p_1 = 0.4$.

Solution to exercise 4.17 (p.85). The Gibbs entropy is $k_{\rm B}\sum_i p_i \ln \frac{1}{p_i}$, where i runs over all states of the system. This entropy is equivalent (apart from the factor of $k_{\rm B}$) to the Shannon entropy of the ensemble.

Whereas the Gibbs entropy can be defined for any ensemble, the Boltzmann entropy is only defined for *microcanonical* ensembles, which have a probability distribution that is uniform over a set of accessible states. The Boltzmann entropy is defined to be $S_{\rm B} = k_{\rm B} \ln \Omega$ where Ω is the number of accessible states of the microcanonical ensemble. This is equivalent (apart from the factor of $k_{\rm B}$) to the perfect information content H_0 of that constrained ensemble. The Gibbs entropy of a microcanonical ensemble is trivially equal to the Boltzmann entropy.

We now consider a thermal distribution (the *canonical* ensemble), where the probability of a state $\mathbf{x}$ is

$$P(\mathbf{x}) = \frac{1}{Z} \exp\left(-\frac{E(\mathbf{x})}{k_{\rm B}T}\right). \tag{4.50}$$

With this canonical ensemble we can associate a corresponding microcanonical ensemble, an ensemble with total energy fixed to the mean energy of the canonical ensemble (fixed to within some precision ϵ). Now, fixing the total energy to a precision ϵ is equivalent to fixing the value of $\ln {}^{1}\!/\!P(\mathbf{x})$ to within $\epsilon k_{\rm B}T$. Our definition of the typical set $T_{N\beta}$ was precisely that it consisted of all elements that have a value of $\log P(\mathbf{x})$ very close to the mean value of $\log P(\mathbf{x})$ under the canonical ensemble, $-NH(X)$. Thus the microcanonical ensemble is equivalent to a uniform distribution over the typical set of the canonical ensemble.

Our proof of the 'asymptotic equipartition' principle thus proves – for the case of a system whose energy is separable into a sum of independent terms – that the Boltzmann entropy of the microcanonical ensemble is very close (for large N) to the Gibbs entropy of the canonical ensemble, if the energy of the microcanonical ensemble is constrained to equal the mean energy of the canonical ensemble.

Solution to exercise 4.18 (p.85). The normalizing constant of the Cauchy distribution

$$P(x) = \frac{1}{Z}\frac{1}{x^2+1}$$

is

$$Z = \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x^2+1} = \left[\tan^{-1} x\right]_{-\infty}^{\infty} = \frac{\pi}{2} - \frac{-\pi}{2} = \pi. \tag{4.51}$$

The mean and variance of this distribution are both undefined. (The distribution is symmetrical about zero, but this does not imply that its mean is zero. The mean is the value of a divergent integral.) The sum $z = x_1 + x_2$, where x_1 and x_2 both have Cauchy distributions, has probability density given by the convolution

$$P(z) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}x_1 \, \frac{1}{x_1^2+1}\frac{1}{(z-x_1)^2+1}, \tag{4.52}$$

which after a considerable labour using standard methods gives

$$P(z) = \frac{1}{\pi^2} 2 \frac{\pi}{z^2+4} = \frac{2}{\pi}\frac{1}{z^2+2^2}, \tag{4.53}$$

which we recognize as a Cauchy distribution with width parameter 2 (where the original distribution has width parameter 1). This implies that the mean of the two points, $\bar{x} = (x_1+x_2)/2 = z/2$, has a Cauchy distribution with width parameter 1. Generalizing, the mean of N samples from a Cauchy distribution is Cauchy-distributed with the *same parameters* as the individual samples. The probability distribution of the mean does *not* become narrower as $1/\sqrt{N}$.

The central-limit theorem does not apply to the Cauchy distribution, because it does not have a finite variance.

An alternative neat method for getting to equation (4.53) makes use of the Fourier transform of the Cauchy distribution, which is a biexponential $e^{-|\omega|}$. Convolution in real space corresponds to multiplication in Fourier space, so the Fourier transform of z is simply $e^{-|2\omega|}$. Reversing the transform, we obtain equation (4.53).

Solution to exercise 4.20 (p.86). The function $f(x)$ has inverse function

$$g(y) = y^{1/y}. \tag{4.54}$$

Note

$$\log g(y) = 1/y \log y. \tag{4.55}$$

I obtained a tentative graph of $f(x)$ by plotting $g(y)$ with y along the vertical axis and $g(y)$ along the horizontal axis. The resulting graph suggests that $f(x)$ is single valued for $x \in (0, 1)$, and looks surprisingly well-behaved and ordinary; for $x \in (1, e^{1/e})$, $f(x)$ is two-valued. $f(\sqrt{2})$ is equal both to 2 and 4. For $x > e^{1/e}$ (which is about 1.44), $f(x)$ is infinite. However, it might be argued that this approach to sketching $f(x)$ is only partly valid, if we define f as the limit of the sequence of functions x, x^x, $x^{x^x}, \ldots$; this sequence does not have a limit for $0 \leq x \leq (1/e)^e \simeq 0.07$ on account of a pitchfork bifurcation at $x = (1/e)^e$; and for $x \in (1, e^{1/e})$, the sequence's limit is single-valued – the lower of the two values sketched in the figure.

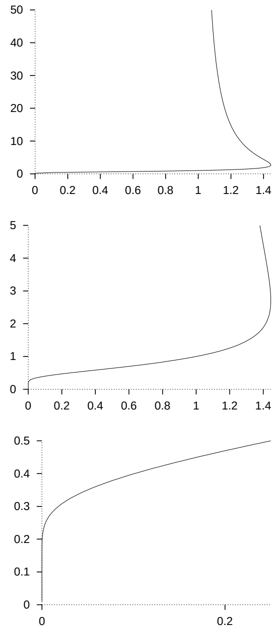


Figure 4.15. $f(x) = x^{x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}}$, shown at three different scales.