Punctured and Irregular High-Rate Gallager Codes

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Abstract

Gallager codes with rates about 0.9 and block length about 2000 or 4000 bits have previously been shown to have promising performance (MacKay and Davey, 2000). In this paper we investigate puncturing as a possible method for further increasing the rate of such codes.

At low signal-to-noise ratios, high-rate punctured Gallager codes appear to be inferior to unpunctured Gallager codes constructed 'beyond the Steiner limit'. We find evidence, however, that the punctured codes are superior at higher signal-to-noise ratios.

We also explore the possible benefits of irregular constructions. For these rates and blocklengths, it seems hard to beat regular Gallager codes.

Keywords: Error-correcting codes, Sum-product algorithm, Magnetic recording.

1 Introduction

A regular Gallager code (Gallager, 1962) has a parity check matrix with uniform column weight j and uniform row weight k, both of which are very small compared to the blocklength. If the code has transmitted blocklength N and rate R then the parity check matrix $\mathbf{H}$ has N columns and M rows, where $M \geq N(1-R)$. [Normally parity check matrices have $M = N(1-R)$, but the matrices we construct may have a few redundant rows so that their rate could be a little higher than $1 - M/N$.]

In an earlier paper we found evidence that regular Gallager codes are useful for high rates ($R > 2/3$) and small block lengths ($N < 5000$) (MacKay and Davey, 2000). Both binary codes and codes over $GF(16)$ had promising performance. These positive results have received further confirmation from studies of magnetic recording applications at IBM Zürich research laboratories (Eleftheriou, 2000).

When constructing Gallager codes it is common practice to impose the constraint that no two columns in the parity check matrix may have an overlap greater than one, in order to reduce the probability of the code's having either low-weight codewords or 'near-codewords'. This constraint, which we will call 'the overlap constraint', imposes a maximum rate on the Gallager code. In the case of regular Gallager codes with column-weight j, the overlap constraint's implications are spelled out in (MacKay and Davey, 2000): for a given blocklength N, the rate R cannot exceed a rate called the Steiner limit, defined implicitly by the equation

$$N(M, j) = \frac{M(M-1)}{j(j-1)}. \qquad (1)$$

For example, if the column weight is $j = 4$, the Steiner limit is 0.9 at a blocklength of $N = 1000$ bits, and 0.94 at $N = 4000$ bits.

In this paper we explore puncturing as a method for further increasing the rate of high-rate Gallager codes. In a punctured code, some of the bits in the graph are omitted from the transmitted codeword. The number of constraints is unchanged, but the transmitted codeword is shorter, so the rate is greater.

Punctured Gallager codes may still be decoded by the sum-product algorithm; the punctured bits are included in the graph, and are inferred during the decoding.

Punctured Gallager codes have been studied before, though not with high rate: linear MN codes (MacKay and Neal, 1995) are punctured Gallager codes in which K bits are punctured, K being the number of source bits. In the parameter regimes we explored, we did not find that such punctured codes were superior to unpunctured Gallager codes.

In this paper two issues are investigated. First, we construct Gallager codes and measure their performance when punctured, comparing them with ordinary Gallager codes, including codes that achieve high rates by violating the overlap constraint. Second, we explore the option of irregular code constructions, which for large blocklength N are known to give superior performance to regular constructions. Do irregular constructions help for high rate codes with $N \simeq 2000$–4000?

2 Method and results

2.1 Increasing rate by puncturing

2.1.1 $N \simeq 2000$. Starting from a four-cycle-free regular $(j, k) = (4, 36)$ Gallager code with N, M, R = 1998, 222, 8/9, we increased the rate to 0.906, 0.914 and 0.923 in two ways:

1. by puncturing 37, 55, and 74 bits from the original code;
2. by making new regular codes violating the overlap constraint. (For each new rate, two codes were selected from a large number of random codes; the selected ones had the smallest number of overlaps.)

Figure 1 compares the empirical performance (block error probability) of these codes, on a Gaussian channel. In all three cases, the punctured Gallager code is inferior to the regular Gallager code on the low signal-to-noise side of the curve. For the cases where 55 and 74 bits are punctured, however, the regular Gallager code shows some evidence of flattening at high signal-to-noise, so it seems that puncturing might be a superior method if performance at high signal-to-noise is important.

In this experiment, none of the codes made any undetected errors.

$N \simeq 4400$. We made a similar experiment with a regular j, N, M, R = 4, 4376, 282, 0.9356 Gallager code. By puncturing 40 and 80 bits, we increased the rate to 0.9442 and 0.9530 respectively. These two punctured codes were each compared with two new Gallager codes violating the overlap constraint. The results, shown in figure 2, are similar to those of figure 1: puncturing is an inferior method at low signal-to-noise, but at the highest rate, the regular Gallager codes with overlaps have an error floor, and the punctured code is superior.

In this experiment, both of the regular rate-0.9530 Gallager codes with overlaps made undetected errors which dominated the error floor.

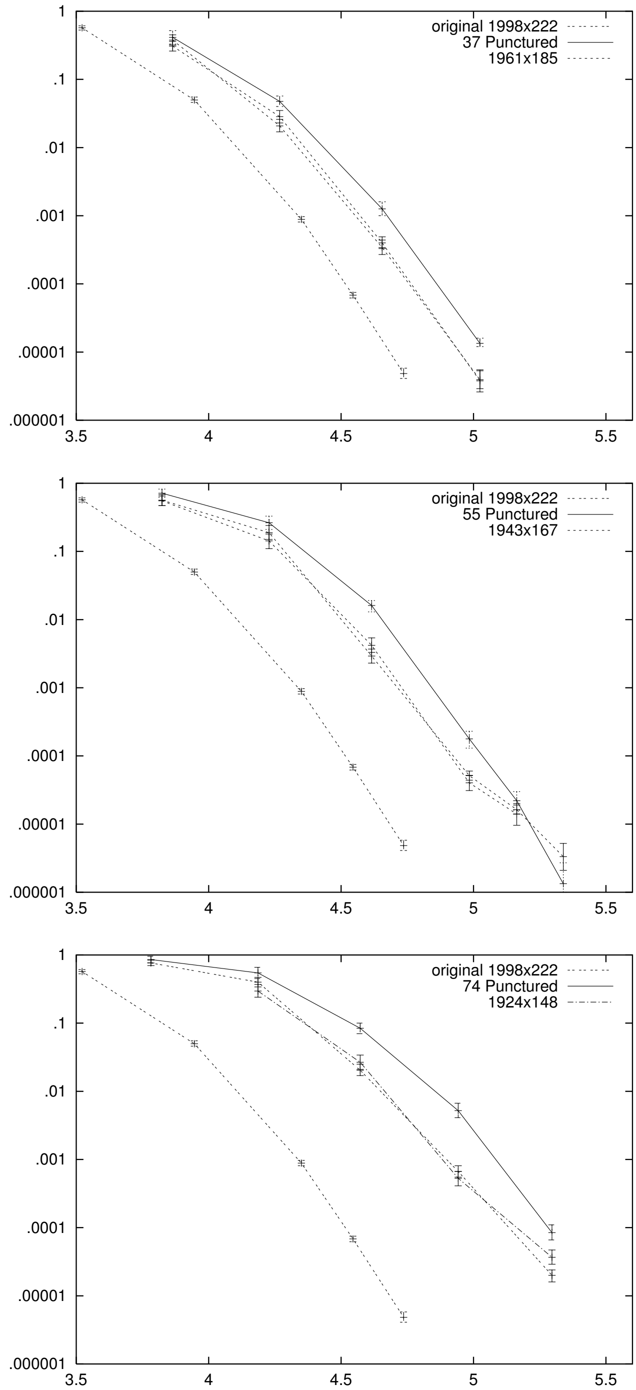

Figure 1: Puncturing compared with violating the overlap constraint as a method for increasing the rate of Gallager codes. From top to bottom, the number of punctured bits is 34, 55, 74. The parity check matrix of the original regular Gallager code with rate $R = 8/9$ and $N = 1998$, `s2.94.594`, can be found in the online archive (MacKay, 1999).

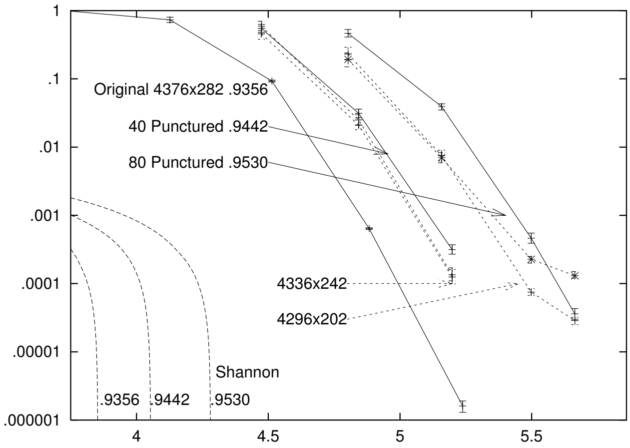

Figure 2: Puncturing comapred with violating the overlap constraint, $N \simeq 4400$, $R \simeq 0.95$. The lower-left curves show the Shannon limits for the three rates studied.

2.2 Irregular high-rate Gallager codes

We compared the $j, N, M = 4, 1998, 222$ regular Gallager code with two irregular codes that were constructed with the aid of Chung *et al.*'s (1999) code design applet. An unconstrained optimization was not used, because that would have led to large numbers of weight-two columns, which we know would produce low-weight codewords. We constrained the minimum column degree to be 4 and found that if columns of degree 14 were permitted, the optimal fraction of such columns was roughly 6% [optimal according to the threshold returned by density evolution]. The predicted improvement in threshold was very small (less than 0.05dB). Two irregular codes of rate 8/9 were constructed having 100 columns of weight 14 and 1898 of weight 4.

Figure 3 shows the empirical results. The irregular codes are scarcely any better at low signal-to-noise, and at high signal-to-noise they have an error floor. These irregular codes did not have any undetected errors.

3 Conclusions

Puncturing is not the best way to make higher rate codes, except possibly at high signal to noise ratios, where punctured codes sometimes seemed to have better slope than ordinary Gallager codes.

It seems difficult to get a benefit from irregular constructions for the rates and block lengths studied here.

Acknowledgements

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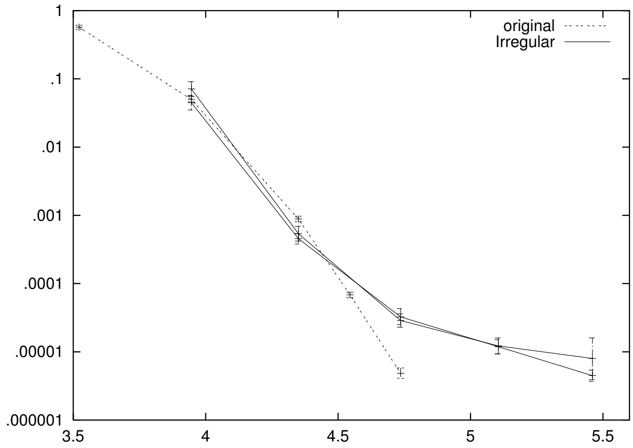

Figure 3: Regular Gallager code compared with irregular.

References

Chung, S.-Y., Urbanke, R. L., and Richardson, T. J., (1999) Ldpc code design applet. `http://truth.mit.edu/~sychung/gaopt.html`.

Eleftheriou, E., (2000) Personal communication. IBM Zürich Research Laboratories.

Gallager, R. G. (1962) Low density parity check codes. *IRE Trans. Info. Theory* **IT-8**: 21–28.

MacKay, D. J. C., (1999) Encyclopedia of sparse graph codes (hypertext archive). `http://wol.ra.phy.cam.ac.uk/mackay/codes/data.ht`

MacKay, D. J. C., and Davey, M. C. (2000) Evaluation of Gallager codes for short block length and high rate applications. In *Codes, Systems and Graphical Models*, ed. by B. Marcus and J. Rosenthal, volume 123 of *IMA Volumes in Mathematics and its Applications*, pp. 113–130. New York: Springer-Verlag.

MacKay, D. J. C., and Neal, R. M. (1995) Good codes based on very sparse matrices. In *Cryptography and Coding. 5th IMA Conference*, ed. by C. Boyd, number 1025 in Lecture Notes in Computer Science, pp. 100–111. Berlin: Springer.