Thresholds of Low-Density Parity Check Codes (Discussion Document)

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Abstract

Does the replica method give correct thresholds for low-density parity-check codes?

This note shows a graph comparing the thresholds derived by the typical set method (McEliece et al), the density evolution method (Richardson and Urbanke), and the replica method (Nakamura and Kabashima).

The typical set method is believed to give an accurate threshold below which the maximum likelihood decoder's *block* error probability tends to zero. [The typical set calculation can also find a threshold below which the *bit* error probability tends to zero. In the case of LDPCCs with $j = 2$, this is what is computed; the average block error probability in this case does not vanish for any noise level.]

The density evolution method is believed to give an accurate threshold below which the sum-product decoder's *bit* error probability tends to zero.

The replica method aims to find two thresholds: p_2, the threshold below which the the maximum likelihood decoder's *bit* error probability tends to zero; and a smaller threshold p_1, a noise level above which the maximum likelihood answer is hard to find, so that the sum-product algorithm is expected to fail.

If the above claims are correct, then we expect $p_1 \geq p_{\rm DE}$.

If the maximum likelihood decoder's bit error probability and block error probability are closely related (as is expected to be the case for a good code), we also expect $p_2 = p_{\rm TS}$.

In the case of bad codes with low-weight codewords, it is possible that the thresholds for bit error probability and block error probability might be different; this is an open question.

Comments on the figure

Codes with good distance Figure 1(a) ($j \geq 3$) has a contradiction: p_2 exceeds $p_{\rm TS}$. So either p_2 is too high, or the typical set method is inaccurate. The replica method's p_1 appears to always lie very close to, and slightly above, the density evolution answer. (The difference is within the numerical precision of Nakamura's calculations.)

Now, is the reason $p_2 > p_{\rm TS}$ because the threshold where the *bit* error probability vanishes is different from the threshold where the *block* error probability vanishes? In the case of a code with bad distance properties, such as the codes with $j = 2$ discussed below, it is conceivable that the two thresholds might be different. But for a code with good distance properties, the thresholds must be identical. This theorem is proved in MacKay (2000).

Theorem 1 *If a sequence of codes with increasing blocklength N has minimum distance satisfying $d_{\rm min} \geq \delta N$, where $\delta > 0$, then, for any given channel, the codeword bit error probability of the optimal bitwise decoder, P_b (also known as the magnetization), and the block error probability of the maximum likelihood decoder, P_B, are related by:*

$$P_B \geq P_b \geq \frac{\delta}{2} P_B. \tag{1}$$

Thus if P_b vanishes then P_B must vanish also, and vice versa.

So the bit and block *thresholds* of a code with good distance are identical.

Codes with bad distance Figure 1(b) ($j = 2$) shows similar differences between the methods, but we cannot be sure whether there is a contradiction before we resolve the question whether the bit and block error probabilities are closely related for bad codes like these.

Other comments

I (DJCM) am pretty sure that the replica method sometimes gives incorrect results. In particular, the claim that MN codes with row weight greater than 2 achieve the Shannon limit, given an optimal decoder, seems incorrect to me, given that we have typical set decoder results that do not achieve the Shannon limit. Strictly speaking, the typical set results are inequalities, so they cannot disprove the replica claim, but I am very sceptical indeed.

Acknowledgements

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References

MacKay, D. J. C., (2000) On thresholds of codes. wol.ra.phy.cam.ac.uk/mackay/abstracts/theorems.html

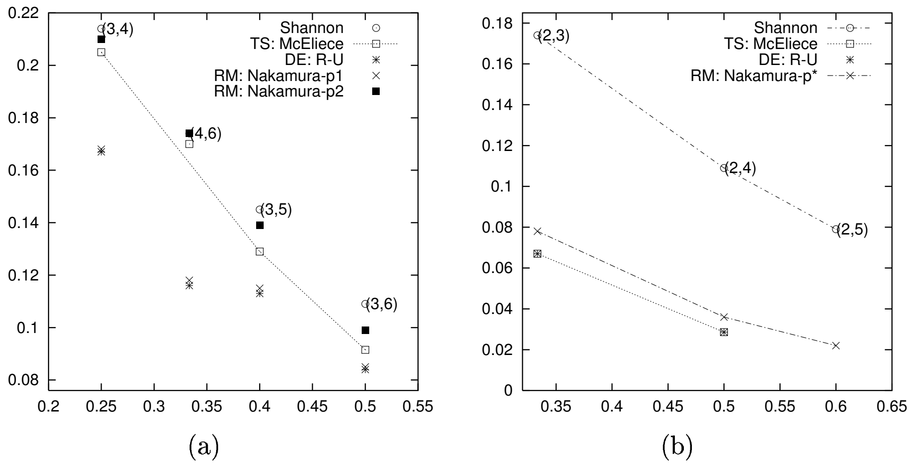

Figure 1: (a) $j = 3$ and $j = 4$. (b) $j = 2$.