3 Dimensional Analysis

This chapter teaches a method of deriving physical laws and of coping with complicated equations without solving them: **dimensional analysis**. Because dimensional analysis is faster than finding an honest solution, you can discard unpromising approaches early, sparing you time to invent promising alternatives. We illustrate the method with examples.

3.1 Geometry

We begin by deriving results in a familiar area: geometry. First, the area of a circle:

$$A = \pi r^2. \tag{3.1}$$

For the sake of learning dimensional analysis, pretend that you forget this formula and begin with what you remember: that the area A depends on the radius r. No other quantities affect the area – at least, we cannot think of any others – so A is a function only of r:

$$A = f(r). \tag{3.2}$$

What is the function f? In the standard way that dimensional analysis is taught, you would note that both sides of the equation must have identical **dimensions**. The left side is a squared length:

$$[A] = \mathrm{L}^2, \tag{3.3}$$

where the brackets indicate 'dimensions of' and L stands for a length. Therefore $[f(r)]$ must also be L^2. Since the only variable in the problem is r and it has dimensions of length, the only functions that satisfy $[f(r)] = \mathrm{L}^2$ are

$$f(r) = \beta r^2, \tag{3.4}$$

where β is a **dimensionless constant**. Being dimensionless, its value is independent of the units that we choose for length – for example, meters or furlongs or light-minutes. Being constant, its value is independent of the circle's radius. Since only radius distinguishes one circle from another, β has the same value for all circles! The Greeks too were surprised by this conclusion. This constant had shown up in related geometry problems and had a name: π. Dimensional analysis does not compute π, but it says that **if you compute it for one circle, then you know it for all circles.** In other words, it tells you a striking fact: that π is a universal constant. It is so important that is deemed worthy of a scarce, non-renewable resource, a letter of the Greek alphabet.

Although dimensional analysis does not produce a value for π, we can do experiments to find π. Figure 3.1 shows a circle. The coarse grid in Figure 3.2 helps estimate the area: If $r = 1$, so each square has side length 1, then $A \approx 4$, which means $\pi \approx 4$. This method overestimates π, but it is a useful overestimate when making approximations. Geometrically it means replacing circles by squares. That replacement makes some problems easier. In other problems, for example with painful boundary conditions in differential equations, squares might produce more unpleasant mathematics than circles do, so you might replace squares with circles, which explains the title of the classic text on estimation in environmental modeling: *Consider a Spherical Cow* [21].

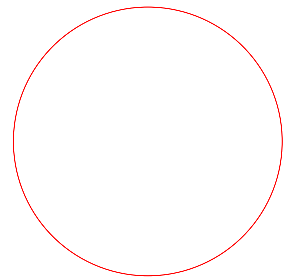

Figure 3.1. *Circle of unit radius.*

You can increase the accuracy of the estimate by measuring the area with a finer grid, as in Figure 3.3, or by using clever methods, such as the relation between circumference and radius (the Greek method). But the important point is the universality of π. Even without a theory to compute π, you can do an experiment on one circle and thereby measure it for all circles. This consequence of dimensional analysis may seem mundane for the area of a circle, but that's because we easily forget how striking the universality of π is. We will use dimensional analysis to find universal conclusions in less familiar domains, including pendulum motion (Section 3.3) and fluid drag (Section 4.1).

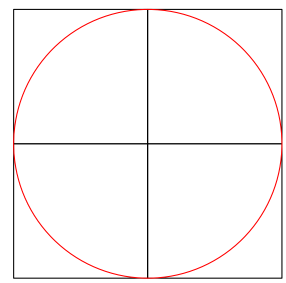

Figure 3.2. *Circle of unit radius with grid for estimating π.*

A slightly more complicated problem, which reuses these ideas and introduces new ones, is the area of an ellipse (Figure 3.4). An ellipse has two radii: the semimajor axis a and the semiminor axis b. So the variables for finding the area are A, a, and b:

$$A = f(a, b), \tag{3.5}$$

where the function $f(a, b)$, this time of two variables, has dimensions of L^2. Now, continuing with the usual method, you look for combinations $a^\alpha b^\beta$ with dimensions of L^2. Any combination with $\alpha + \beta = 2$ has these dimensions:

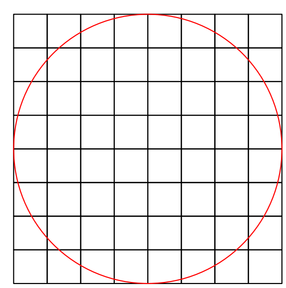

Figure 3.3. *Circle of unit radius with fine grid. The fine grid produces a more accurate estimate of π than does the coarse grid in Figure 3.2.*

$$a^3/b, \quad ab, \quad b^2, \quad \text{and } a^2, \tag{3.6}$$

are a few members of this infinite set. The area could be any (or none) of them. The disaster is not over, however. Sums of these combinations also have dimensions of L^2, so the following possibilities might be correct:

$$A = \begin{cases} ab + b^2, \\ b^2 + a^3/b, \\ a^2 - b^2, \\ \vdots \end{cases} \tag{3.7}$$

The general formula is then

$$A = \sum_{\alpha} f_\alpha a^\alpha b^{2-\alpha}, \tag{3.8}$$

where f_α is the coefficient of the term with a^α in it. This representation is not a fruitful way to think about area. Suppose, for example, that the correct function is $A = (a^2 + b^2)/(a + b)$. This relatively compact form turns into an infinite series in $a^\alpha b^{2-\alpha}$. Deducing the compact form from the infinite series (3.8) is not easy. The usual method of dimensional analysis has broken down. The area of a circle is a simple-enough problem that the flaws in the method did not appear. However, handling an extra variable (since an ellipse has two radii) exposes its serious flaw.

So we redo the circle example with a reliable, insightful method and then return to the ellipse. For the circle the variables were A and r, so $A = f(r)$ as we derived in (3.2). Both sides have dimensions of L^2 – and here we add the important new step – so **make each side dimensionless** by dividing by r^2:

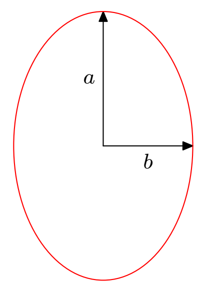

Figure 3.4. Ellipse.

$$\frac{A}{r^2} = \text{dimensionless quantity.} \tag{3.9}$$

The left side is a **dimensionless group**: *dimensionless*, because it has no dimensions; and *group* because it combines one or more variables from the problem. Being dimensionless it can be written in terms of dimensionless quantities.

Dimensionless quantities are the building blocks of physical law. Units – whether meters, feet, seconds, pints, fortnights, or furlongs – are artificial. The universe does not cares about our choice of units: Physical laws, such as $E = mc^2$, take the same form in every system of units. Since only dimensionless quantities – pure numbers – are the same in every unit system, dimensionless quantities are the natural representation for physical laws. We therefore write equations in universe-friendly, dimensionless form.

In this problem, the only dimensionless quantities are constants such as 2, π, or e^2, as well as any dimensionless groups. Let's ignore the constants for now and find the dimensionless groups. To find dimensionless groups, tabulate information about the variables (Table 3.1). Making a table, although not necessary in this example, is a useful habit for complex problems with many variables. Now you have two choices. First, you can set up and solve linear equations to find how to combine the variables in Table 3.1. A dimensionless group is a combination $G = A^\alpha r^\beta$, with α and β to be found, and where G has no powers of length, mass, or time. Since A has two powers of length and r has one power of length, the requirement for G to have no powers of length produces this equation:

$$0 = 2\alpha + \beta. \tag{3.10}$$

Since A and r have no mass or time in them, we get no equations for the mass or time dimensions. The solution to (3.10) is $\beta = -2\alpha$, so

$$G = A^{\alpha} r^{-2\alpha} = \left(\frac{A}{r^2}\right)^{\alpha}. \tag{3.11}$$

Another method, which works fine in this case, is to look at the list and think hard. You might find groups such as:

$$\frac{A}{r^2}, \quad e^{A/r^2}, \quad \sqrt{r^2/A}, \quad \text{and } \ln\frac{A}{r^2}. \tag{3.12}$$

Or you can use a slightly more organized method of finding groups, whose reasoning goes as follows:

> Hmm, we want a group with no dimensions. Let's start with one of the variables, say A. It has two lengths in it. How can we get rid of the lengths? Are there other variables with just a length? Yes, r! So we can use r twice to cancel the lengths in A. Thus we form A/r^2. Oh, and any function of it, such as $\ln(A/r^2)$ or e^{A/r^2}.

This more organized method here seems hardly an improvement over just looking at the list, but it will shine when we solve more complicated problems, and be more useful than the linear-equations method.

For now never mind how we got the possible groups and instead study the list of possibilities. The list is redundant: From almost any member, say A/r^2, we can compute the others. So the two variables A and r produce one **independent** dimensionless group. Any statement about the area of a circle can be written as

$$\frac{A}{r^2} = \text{dimensionless quantity}. \tag{3.13}$$

The dimensionless quantity on the right must be formed from dimensionless constants and dimensionless groups. Perhaps then

$$\frac{A}{r^2} = g\left(\frac{A}{r^2}, \text{dimensionless constants}\right)? \tag{3.14}$$

That form is of little use. We want to find A, not get an equation with A on both sides. However, A/r^2 is the only independent dimensionless group, so taking it away from the right side leaves only

$$\frac{A}{r^2} = \text{dimensionless constant}, \tag{3.15}$$

or

$$A = \text{dimensionless constant} \times r^2. \tag{3.16}$$

This result is familiar, with the dimensionless constant being π. The pattern is to rewrite the problem as:

$$\text{group containing } A = g\,(\text{other groups}). \tag{3.17}$$

Dimensionless constants of proportionality recur in this textbook; often their value is not important and their presence clutters equations and clouds thinking. To clarify our thinking, we often use the $\sim$ notation:

$$A \sim r^2 \tag{3.18}$$

to say that A and r^2 share the same dimensions but differ by a dimensionless factor.

Now we redo the ellipse with this new method. Its three variables are tabulated in Table 3.2. The first step is to rewrite $A = f(a, b)$ in the dimensionless pattern of (3.17). To that end, we find independent dimensionless groups formed from A, a, and b. Again the search is easy (which is why we introduce the method in these geometry problems) because all the variables are built from one dimension, length. Dimensionless groups include:

Var.	*Dimen.*	*What*
A	L^2	area of circle
r	L	radius

Table 3.1. *Variables that might determine the area of a circle.*

$$\frac{A}{ab}, \quad \frac{a}{b}, \quad \frac{A}{b^2}, \qquad \text{and} \qquad \frac{Aa}{b^3}. \tag{3.19}$$

This list is also redundant. For example, the second and third groups multiply to give the fourth group. By trial and error, you can convince yourself that you can make any group from, say,

$$\frac{A}{ab}, \qquad \text{and} \qquad \frac{A}{a^2}. \qquad (3.20).$$

This set, however, will cause problems when we write the result according to the pattern (3.17) because A will appear on both sides. So keep looking for another set of dimensionless groups. Perhaps

Var.	*Dimen.*	*What*
A	L^2	area of ellipse
a	L	semi-major axis
b	L	semi-minor axis

Table 3.2. *Variables that might determine the area of an ellipse.*

$$\frac{A}{a^2}, \qquad \text{and} \qquad \frac{a}{b}? \tag{3.21}$$

This set avoids the problem of A appearing on both sides. However, it has its own problem. Most properties about ellipses do not care which length is a and which length is b; and the combination A/a^2 should, but does not, respect this symmetry. Instead let's try A/ab. This choice leads to the following set of groups:

$$\frac{A}{ab} \qquad \text{and} \qquad \frac{a}{b}. \tag{3.22}$$

Then, following the pattern (3.17), you get

$$\frac{A}{ab} = f\left(\frac{a}{b}\right). \tag{3.23}$$

This right side, with its unknown function f, is more complicated than its counterpart (3.15) in the circle problem.

We need to find the function f. Many arguments give us clues about its form. For example, symmetry again: The area remains the same if the ellipse rotates 90°. This rotation interchanges a and b (Figure 3.6), so interchange a and b in (3.23):

$$\frac{A}{ba} = f\left(\frac{b}{a}\right). \tag{3.24}$$

The left sides of (3.23) and (3.24) are identical, so $f(a/b) = f(b/a)$. In terms of $x \equiv a/b$, the function f must satisfy $f(x) = f(1/x)$. One such function is $f(x) = \text{constant}$. Is it the only one? It's not the only function, when considering the full list of functions in the world. But for the ellipse problem, it might be the only function. To decide, try a thought experiment: Imagine doubling a and therefore x. This change doubles the area (Figure 3.5), so A/ab remains unchanged and so should $f(x)$. This argument would be true when tripling a, or making any change to a, and therefore to x. So $f(x)$ is independent of x, which means it is a constant. Therefore

$$\frac{A}{ab} = \text{dimensionless constant}. \tag{3.25}$$

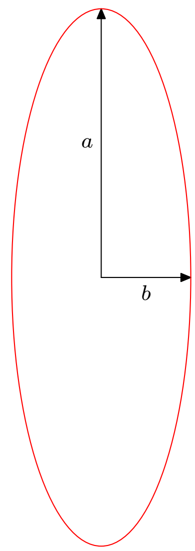

Figure 3.5. *Doubling the semimajor axis doubles the area.*

As in the circle problem, the constant is universal: Every ellipse has the same constant. You could experiment with an ellipse to measure it, or you can do another thought experiment: Imagine a **limiting case** of an ellipse. The simplest limiting case, where $b \to 0$, is a line and is not helpful. But the case where $b \to a$ turns an ellipse into a circle, whose behavior we understand. Changing b does not change the dimensionless constant in the solution (3.25) – that's what being a universal constant means. For a circle the constant is π, so for an ellipse

$$\frac{A}{ab} = \pi. \tag{3.26}$$

Reasoning about f is easier than deducing an infinite set of coefficients (3.26) that arise in the usual method. The function f is a natural representation for our knowledge (or lack of it). Choosing productive and compact representations is essential to the art of thinking and problem solving.

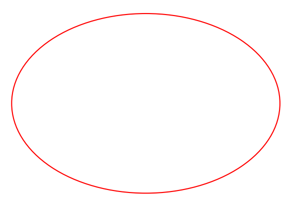
Figure 3.6. *Ellipse rotated 90. This rotation interchanges a with b but preserves the area.*

From the examples of a circle and an ellipse, a few morals arise:

- Write problems and results in dimensionless form. Therefore find dimensionless groups.
- Dimensionless constants are universal.
- Even without a theory, you can approximate dimensionless constants by crude experiments.
- Don't use the usual, linear-equations method of dimensional analysis. Use methods with representations that compactly describe your knowledge and allow you to reason about the problem.

3.2 Pulley

Now let's practice dimensional analysis in a physical problem: masses on a pulley. Many of the ideas and methods that you learned in the geometry example transfer to this problem, and it introduces more methods and ways of reasoning. So, two masses, m_1 and m_2, are connected and thanks to a pulley are free to move up and down (Figure 3.7). What is the acceleration of the masses and the tension in the string? You can solve this problem with standard methods from first-year physics, which means that you can can check the solution that we derive using dimensional analysis, educated guessing, and a feel for functions.

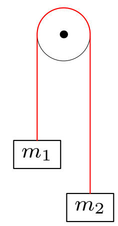

Figure 3.7. *Pulley.*

The first problem is to find the acceleration of, say, m_1. Since m_1 and m_2 are connected by a rope, the acceleration of m_2 is, depending on your sign convention, either equal to m_1 or equal to $-m_1$. Let's call the acceleration a and use dimensional analysis to guess its form. The first step is to decide what variables are relevant. The acceleration depends on gravity, so g should be on the list. The masses affect the acceleration, so m_1 and m_2 are on the list. And that's it (Table 3.3). You might wonder what happened to the tension: Doesn't it affect the acceleration? It does, but it is itself a consequence of m_1, m_2, and g. So adding tension to the list does not add information; it would instead make the dimensional analysis difficult.

These variables fall into two pairs where the variables in each pair have the same dimensions. So there are two dimensionless groups here ripe for picking: $G_1 = m_1/m_2$ and $G_2 = a/g$. You can make any dimensionless group using these two obvious groups, as experimentation will convince you. Then, following the usual pattern,

$$\frac{a}{g} = f\left(\frac{m_1}{m_2}\right), \tag{3.27}$$

where f is a dimensionless function.

Var.	*Dimen.*	*What*
a	LT^{-2}	accel. of m_1
g	LT^{-2}	gravity
m_1	M	block mass
m_2	M	block mass

Table 3.3. *Variables that might determine the acceleration of* m_1.

Pause a moment. The more thinking that you do to choose a clean representation, the less algebra you do later. So rather than find f, let's think about the group m_1/m_2. It does not respect the symmetry of the problem, in that hardly anything should change when you interchange the labels m_1 and m_2 – just as nothing changes about the ellipse area when you interchange the semimajor and semiminor axes labels. By that standard, $G_1 = m_1/m_2$ is not a terrible choice, since a mass interchange takes G_1 to $1/G_1$, which is closely related to G_1. However, it means that G_1 varies from 0 to ∞ and is symmetric about $G_1 = 1$ when you take the its reciprocal. This is not a pleasant symmetry since it maps a finite segment (the range $[0, 1]$) to an infinite segment (the range $[1, \infty]$). One solution is to use a logarithm and choose $G_1 = \ln(m_1/m_2)$. Mass interchange then maps an infinite segment to an infinite segment. A logarithm suggests the presence of exponentials, but falling objects usually have algebraic solutions. So a logarithm seems out of place in this problem.

Back to the drawing board for how to fix G_1. Another option is to use $m_1 - m_2$. Wait, you cry, that is not dimensionless! We will fix that problem in a moment. For now observe its virtues. It shows a physically reasonable symmetry under mass interchange: $G_1 \to -G_1$. And the range of this G_1 is finite. We'd like to preserve those virtues while making G_1 dimensionless. To make it dimensionless, divide it by another mass. One candidate is m_1:

$$G_1 = \frac{m_1 + m_2}{m_1}. \tag{3.28}$$

That choice, like dividing by m_2, abandons the beloved symmetry. But dividing by $m_1 + m_2$ solves all the problems:

$$G_1 = \frac{m_1 - m_2}{m_1 + m_2}. \tag{3.29}$$

This group is even better than $m_1 - m_2$ because its range, $[-1, 1]$, is symmetric and independent of m_1 and m_2, whereas $m_1 - m_2$ has a symmetry point at $(m_1 - m_2)/2$, which is not at the origin. So we have several reasons to like the choice (3.29). Using it the solution pattern (3.27) becomes

$$\frac{a}{g} = f\left(\frac{m_1 - m_2}{m_1 + m_2}\right), \tag{3.30}$$

where f is a dimensionless function, probably a different function from the f in (3.27).

To guess $f(x)$, where $x = G_1$, again try extreme or limiting cases. First imagine m_1 huge. A quantity with mass cannot be huge on its own, however. Here huge means *huge relative to* m_1, whereupon $x \approx 1$. In that experiment, m_1 falls as if there were no m_2 (Figure 3.8) so $a = -g$. Here we've chosen a sign convention with positive acceleration being upward. If m_2 is huge relative to m_1, which means $x = -1$, then m_2 falls like a stone and drags m_1 upward, so $a = g$. A third limiting case is $m_1 = m_2$ or $x = 0$, whereupon the masses are in equilibrium so $a = 0$. Figure 3.9 plots our knowledge of f. A simple conjecture – an educated guess – is that $f(x) = x$. Then we have our result:

$$\frac{a}{g} = \frac{m_1 - m_2}{m_1 + m_2}. \tag{3.31}$$

Now let's apply the same kind of reasoning to find the tension in the string. The variables are the same as above but with a replaced by T (Table 3.4). We reuse the hard-won dimensionless group

$$G_1 = \frac{m_1 - m_2}{m_1 + m_2}. \tag{3.33}$$

The second group must contain T, since it is the quantity for which we want to solve. To cancel the dimensions of T, we need to create

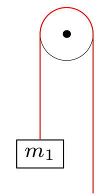

Figure 3.8. *Pulley with $m_1 \gg m_2$.*

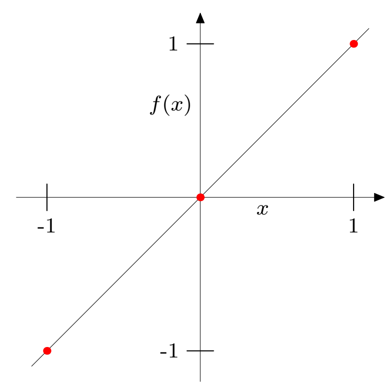

Figure 3.9. *Data about pulley acceleration. Here*

$$x = \frac{m_1 - m_2}{m_1 + m_2} \quad (3.32).$$

Extreme-cases reasoning produces a few data points for $f(x) = a/g$. They are plotted here in dimensionless form (which is why the axes have no units!). The straight line is then a reasonable guess for the f in (3.30).

Var.	*Dimen.*	*What*
T	MLT^{-2}	rope tension
g	LT^{-2}	gravity
m_1	M	block mass
m_2	M	block mass

Table 3.4. *Variables that might determine the acceleration of m_1.*

another force. Two candidates are $m_1 g$ and $m_2 g$ but neither respects the symmetry between m_1 and m_2. Another option is $(m_1 - m_2)g$. It has the defect that combining it with T gives

$$\frac{T}{(m_1 - m_2)g,} \tag{3.34}$$

which goes to infinity when $m_1 = m_2$, whereas the tension itself never misbehaves like that. So let's use

$$G_2 = \frac{T}{(m_1 + m_2)g}.$$

Then the general pattern becomes

$$\frac{T}{(m_1 + m_2)g} = f\left(\frac{m_1 - m_2}{m_1 + m_2}\right). \tag{3.35}$$

As with the acceleration, to find $f(x)$ try limiting cases: $m_1 \gg m_2$, $m_1 \ll m_2$, and $m_1 = m_2$. In the first case, where $x = 1$, mass 1 falls unhindered by mass 2, and the string is relaxed ($T = 0$). In the second case, mass 2 falls unhindered and again $T = 0$. In the third case, the system is in equilibrium and the tension counteracts gravity for each mass, so $T = m_1 g$ and $T = m_2 g$ – which is possible since $m_1 = m_2$. Thus

$$\frac{T}{(m_1 + m_2)g} = \frac{1}{2}.$$

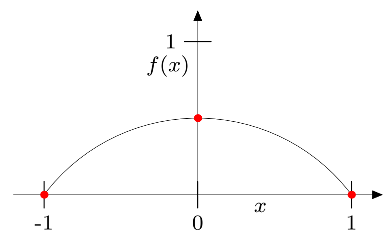

Figure 3.10. *Data about string tension. Extreme-cases reasoning produces a few data points, plotted here in dimensionless form. Since the middle point is at* $(0, 1/2)$*, the parabola* $f(x) = (1 - x^2)/2$ *is a reasonable guess for the* f *in (3.35).*

Figure 3.10 shows the data from these thought experiments. The simplest curve that passes through the points is the parabola $f(x) = (1 - x^2)/2$. You can fit this curve using official methods or using the following mental dialogue:

> A straight line won't do, since these points need a hump in the middle, so how about a curve with a higher power of x? The simplest such curve is a parabola and the standard parabola is $f(x) = x^2$. It, however, points upward instead of downward. To fix that problem, let's try $f(x) = -x^2$. This choice points downward but it does not match the points at $(\pm 1, 0)$. To fix that problem, add 1 to get $f(x) = 1 - x^2$. This choice matches the two outer points; however, the middle point should have $f(x) = 1/2$ rather than $f(x) = 1$. So multiply f by 1/2 to get
>
> $$f(x) = \frac{1 - x^2}{2}. \tag{3.36}$$

The tension is then given by

$$\frac{T}{(m_1 + m_2)g} = \frac{1}{2}\left[1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2\right]. \tag{3.37}$$

After expanding the quantities in parentheses, you get

$$T = g\,\frac{2 m_1 m_2}{m_1 + m_2}. \tag{3.38}$$

In the exercises you can check this result using the standard methods of first-year physics (freebody diagrams and Newton's laws).

3.3 Pendulum: Dimensionless groups

Our next problem, which reuses ideas from the last few problems and introduces new ideas, is the pendulum. A bob hangs at the end of a massless rope; from a resting state the bob starts oscillating (Figure 3.11). What is its period of oscillation? Clocks and seafaring empires once depended on the result, as Matthews describes in a wonderful book about physics, history, and science education [43]. Rather than solve the differential equation describing the motion, we use dimensional analysis.

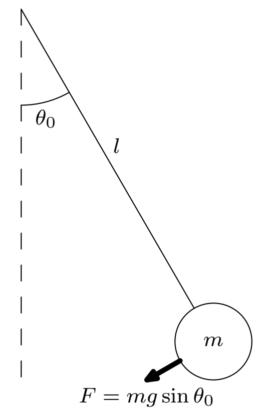

Figure 3.11. *A pendulum bob of mass m hangs from a massless rope of length l. The bob is released from rest at an angle θ_0.*

On what quantities can the period τ depend? Imagine life as a pendulum bob. Why do you move? Because of gravity, so g belongs on the list. A heavier (or more accurately, a more massive) bob feels a stronger gravitational force, so m belongs on the list. You may object that we have personified the pendulum bob, that we have endowed it with the capacity for feeling, or at least, for feeling forces. We have even become a pendulum bob. We plead guilty with an explanation. Picturing an active bob enhances our intuition for how it behaves; to make a vivid picture of an active bob, we pretend to be a bob. Similarly, looking for a lost marble, you might ask yourself, 'If I were a marble and someone dropped me, where would I roll?' To see whether this style enhances your intuition, try it for one or two months.

If you race ahead of our story, you might list the initial angle θ_0. You are right to include it, but pause to apply the principle of *maximal laziness*: Why add complications today that you can postpone to tomorrow? We first squeeze results out of the few quantities τ, g, and m. Table 3.5 tabulates these quantities and their dimensions. No combination of g, m, and τ is dimensionless, as you can show by setting up a few linear equations in the powers of the fundamental dimensions: length, mass, and time. Or you can reason as follows:

Var.	*Dimen.*	*What*
m	M	mass of bob
τ	T	period
g	LT^{-2}	gravity

Table 3.5. *Variables that might determine the oscillation period of a pendulum.*

> Only m contains mass, so it had better not appear in a dimensionless group because if it did, no other variable would be able to cancel the mass. Similarly, g cannot appear in a dimensionless group because it is the only variable containing a length. So only τ is left. And it cannot form a dimensionless group alone.

To make a dimensionless combination, and therefore to write a dimensionless equation, we need at least one more quantity: ideally one containing length to cancel the lengths in g. Length? Ah, long pendula swing slowly, so we should include the length of the string. The relevant variables, with this addition, are listed in Table 3.6.

Try educated trial and error to find the groups. First eliminate the easy dimensions. Only one variable, m, contains mass. To be dimensionless, the group must contain another variable that cancels the mass contributed by m. So m must cancel itself from any group that it tries to join.[1] Only g and l contain length, each linearly, so g and l must enter the dimensionless combination as g/l. Only τ and g

1. *"Any club that would admit me as a member, I wouldn't want to join."*
– Groucho Marx

contain time, so $g\tau^2$ must enter together. Therefore

$$G_1 = \frac{g\tau^2}{l} \tag{3.39}$$

is a dimensionless group. By looking for other groups, you'll convince yourself that they are functions of $g\tau^2/l$. For example:

$$\left(\frac{g\tau^2}{l}\right)^2, \quad \sin\frac{g\tau^2}{l}, \quad \text{or} \quad \exp\frac{g\tau^2}{l}. \tag{3.40}$$

Why do we prefer the $g\tau^2/l$ group, when we could choose any function of it? First, we plan to solve for τ, so we want τ in the numerator of the group. Second, we have little idea whether f has logarithms, sines, cosines, or whatever in it. If we choose $\sin(g\tau^2/l)$ as the group, then f might acquire an annoying arcsin to undo this silly choice. Keep the groups simple!

As in the geometry problems,

$$\frac{g\tau^2}{l} = f(\text{other dimensionless groups}). \tag{3.41}$$

Var.	*Dimen.*	*What*
m	M	mass of bob
τ	T	period
g	LT^{-2}	gravity
l	L	length of bob

Table 3.6. *Variables that might determine the oscillation period of a pendulum.*

Since there are no other independent dimensionless groups, the function f must be a dimensionless constant. Therefore $g\tau^2/l$ is a dimensionless constant. We call the unknown constant Π, in honor of the Buckingham Pi theorem (Theorem 3.1). Then

$$\tau = \sqrt{\Pi}\sqrt{\frac{l}{g}}. \tag{3.42}$$

Dimensional analysis, a mathematical technique, can take us no farther. To make progress, we need to add physics knowledge.

Before making such an effort, spend a moment to check the result. How reasonable is (3.42)? Strong gravity yanks the pendulum hard, decreasing the period. Therefore g should be and is in the denominator. Length should be and is in the numerator, since long pendula swing slowly. Should length appear as a square root? To test that dependence on length, we made a pendulum from a string and weighted it using a full key ring. You can also try a fork, teacup, or giant hex nut. This pendulum completes four periods while we count roughly 6 seconds, saying 'one-one-thousand, two-one-thousand, three-one-thousand, four-one-thousand' for pacing. When we shrink the string by a factor of 4, by folding it in half twice, the pendulum completes four periods in only 3 seconds: one-half of the previous period. The $\sqrt{l}$ in the numerator is therefore plausible. What about g? Those with billions of dollars can test its effect by measuring the period on the moon or on Mars. The rest of us have to have faith in dimensional analysis!

Dimensional analysis does not, and cannot determine Π. Numbers such as Π have no units; they are invisible to dimensional analysis, which cares about only the dimensions of a quantity, not its magnitude. To find Π, you can use data from the experiment above. You can solve differential equations, which we are trying to avoid. Or you can rely on a sly argument due to Huygens (see the exercises). Much of this text assumes (or hopes) that Π is near 1. We often use dimensional analysis to solve problems for which we cannot determine Π; at the end we pretend and hope that $\Pi = 1$. Let's test that assumption by estimating Π.

We can estimate Π using the key ring pendulum. Its string is roughly twice the length of American or A4 paper (which is about one foot long) and the period is roughly 1.5 s, so

$$\Pi = \frac{g\tau^2}{l} \approx \frac{32\,\mathrm{ft\,s^{-2}} \times (1.5\,\mathrm{s})^2}{2\,\mathrm{ft}} \approx 36. \tag{3.43}$$

Did the English units of feet shock you? They should not. One moral of dimensional analysis is that units do not matter. Use any convenient system. If you insist on meters:

$$\Pi = \frac{g\tau^2}{l} \approx \frac{10\,\mathrm{m\,s^{-2}} \times (1.5\,\mathrm{s})^2}{0.6\,\mathrm{m}} \approx 38. \tag{3.44}$$

Whether computed in feet or meters, Π is far from 1! The reason will become clear shortly.

We doubt that the honest physics of the pendulum contains a 36 or 38. Especially not 38, which is 2×19, whereas at least 36 has many factors. If we estimate Π more precisely, we might guess its exact value. So we used a ruler to measure the pendulum length, measuring it from the knot where we hold it to the center of mass of the key ring: $l = 0.65\,\mathrm{m}$ and 10 periods took 15.97 s according to a wrist stopwatch. Then

$$\Pi \approx \frac{9.81\,\mathrm{m\,s^{-2}} \times (1.597\,\mathrm{s})^2}{0.65\,\mathrm{m}} \approx 38.49. \tag{3.45}$$

That value is remarkably close to 40, and π^2 is remarkably close to 10, so perhaps $\Pi = 4\pi^2$, a combination that physics might generate, unlike 36 or 38.

The honest method, which we usually avoid, sets up the pendulum differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0. \tag{3.46}$$

The solutions are $\theta(t) = \cos\omega t$ or $\sin\omega t$, where $\omega = \sqrt{g/l}$. So the period, which is $2\pi/\omega$, is:

$$\tau = 2\pi\sqrt{\frac{g}{l}}. \tag{3.47}$$

The dimensionless constant Π is indeed $4\pi^2$, or about 40, which is hardly close to 1, because we estimated the period instead of the angular frequency. Here you see another moral of dimensional analysis:

Estimate quantities relating to radians rather than to 2π radians (full periods); otherwise your answers might be inaccurate by a power of 2π.

3.4 Pendulum with another variable

We promised to reinstate the release angle θ_0. Despite our earlier salutary message about using ω instead of τ, we continue to use τ so that the results can be easily compared to those without θ_0. Table 3.7 contains the new list. From these five variables made of three dimensions, we can – says the Buckingham Pi theorem (Theorem 3.1) – form two independent dimensionless groups. One group is the previous group (3.39). The second is easy: θ_0 is already dimensionless. The two groups are then

$$\begin{aligned} G_1 &= \theta_0, \\ G_2 &= \frac{g\tau^2}{l}. \end{aligned} \tag{3.48}$$

Var.	Units	Description
θ_0	–	initial angle
m	M	mass of bob
τ	T	period
g	LT^{-2}	gravity
l	L	length of rope

Table 3.7. *Updated variables that may determine a pendulum's oscillation period.*

In terms of the original variables, the result is $g\tau^2/l = f(\theta_0)$, or

$$\tau = f(\theta_0)\sqrt{\frac{l}{g}}. \tag{3.49}$$

We do not know the function $f(\theta_0)$. You can determine it by experiment: Release a pendulum at various θ_0 and measure $\tau(\theta_0)$. Then, f is

$$f(\theta_0) = \tau(\theta_0)\sqrt{\frac{g}{l}}. \tag{3.50}$$

We do not have to repeat the experiments for different l (say, for another pendulum) or g (say, on another planet), because f is a **universal function**, just as π is a universal constant. All pendulums – long or short, on the earth or on Mars – obey the same f.

However, dimensional analysis cannot determine f. We know its behavior in a few simple cases. First, when $\theta_0 \to 0$, then the home experiment in the previous section suggests that $f \to 2\pi$; Huygens's circular-pendulum argument confirms it. The other extreme is releasing the pendulum vertically upward, when $\theta_0 = \pi$. If θ_0 is exactly π then the pendulum (with a steel rod instead of a string!) will hang forever: $\tau = \infty$. Once again we get free data by imagining extreme cases.

How does f behave near $\theta_0 = 0$? Symmetry helps us reason about f: Nature does not care whether we release the pendulum on the left (negative θ_0) or on the right (positive θ_0). So $f(\theta_0)$ is symmetric about the y axis, and its power series cannot contain odd powers of θ_0. It is therefore approximately:

$$1 + \alpha\theta_0^2 + \cdots, \tag{3.51}$$

where α is an unknown dimensionless constant. Whatever the value of α, the interesting conclusion is that f is flat (has no first derivative) near the origin. The result is that a pendulum's period depends only weakly on amplitude, and a pendulum clock is a reliable timepiece even without a mechanism to replace energy lost to friction.

3.5 The pattern of argument

The general pattern of argument that we used in this example is useful in many order-of-magnitude analyses. By dreaming or otherwise, we conjectured a list of relevant quantities: gravity, mass, length, and period (and perhaps the initial angle). Making this list is difficult. Leaving out a necessary variable invites trouble. The first list – gravity, mass, and period – could not form a dimensionless group. This failure is a clue that we had neglected a necessary variable – in this case, the length. We can also make subtle mistakes. Suppose we think that the relevant length is the width of the rope, w. Then $g\tau^2/w$ is the group and the period would be

$$\tau \sim \sqrt{\frac{g}{w}}. \tag{3.52}$$

Although dimensionally correct, this equation is empirically bogus. Make a pendulum using thin fishing line and compare it to one using the same length of twine. They swing with almost exactly the same period. Or imagine two identical pendula swinging alongside one another. If you glue the strings, the thickness doubles but the period will not change. If you are to include all relevant variables, you must think physically, appeal to experiment, and make lucky guesses.

However, do not include every semi-reasonable quantity. Irrelevant variables multiply the possibilities for dimensionless relations. For example, suppose that, to be safe, we include l and w. We can form two dimensionless groups, $g\tau^2/l$ and w/l. The period then satisfies

$$\frac{g\tau^2}{l} = f\left(\frac{w}{l}\right). \tag{3.53}$$

Physics knowledge now restricts the form of f. You know that the width of the string is irrelevant; then we recover the simpler relation $g\tau^2/l =$ constant derived before but we do more work to get there. We know of no recipe for choosing the right set of variables, except to practice. Once we chose the variables, we found the only dimensionless group (apart from transformations): $g\tau^2/l$. Then $\tau = \sqrt{\Pi l/g}$. The unknown constant Π we determined by experiment (or you can use a the argument of Huygens).

3.6 Generalizing the argument: The Buckingham Pi theorem

One art of dimensional analysis lies in choosing the set of relevant variables; a computer would find this stage difficult, if not impossible.

The part that we could program – finding the dimensionless groups – we normally do by guessing. But searching for the groups is made easier if you know how many to look for. That is what the following theorem tells you.

Theorem 3.1 (Buckingham Pi theorem) A system described by n variables, built from r independent dimensions, is also described by $n-r$ independent dimensionless groups.

3.7 What you have learned

Every valid physical equation can be written in a form without units. To find such forms, follow these steps:

1. Write down – by magic, intuition, or luck – the physically relevant variables. For illustration, let's say that there are n of them.
2. Determine the units of each variable. Count how many independent dimensions these variables comprise. Call this number r. Usually, length, mass, and time are all that you need, so $r = 3$.
3. By playing around, or by guessing with inspiration, find $n - r$ independent dimensionless combinations of the variables. These combinations are the dimensionless groups, or Pi variables, named after the Buckingham Pi theorem.
4. Write down the result in the form

$$\text{one group} = f(\text{other groups}). \tag{3.54}$$

   Using physical arguments to eliminate dimensionless groups or to restrict the form of f. Don't be afraid to guess and conjecture.

In the next chapter we apply this method to fluid mechanics: to drag and its consequences.

3.8 Exercises

► **3.13** *Check*
Verify the acceleration (3.31) and tension (3.38) derived in the pulley example.

► **3.14** *Kepler 3*
For circular orbits, use dimensional analysis to derive Kepler's third law for the period.

► **3.15** *Huygens' method*
Huygens invented a sly method to find the dimensionless constant for the period of a pendulum. Imagine a conical pendulum: a pendulum moving in a circle of radius $r = l \sin\theta$, where l is the length of the string and θ is the angle that it makes with the vertical (θ stays constant as the pendulum moves in a circle). Continue the thought

experiment to show that, for small θ:

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{3.55}$$

Don't solve any differential equations!

► **3.16** *Black holes*
Use dimensional analysis to estimate the radius of a black hole (for the curious, you are estimating the radius of the event horizon). What is this radius for an object with the mass of the sun?