4 Fluid drag

Pendulum motion is not a horrible enough problem to show the full benefit of dimensional analysis. Instead try fluid mechanics – a subject notorious for its mathematical and physical complexity; Chandrasekhar's books [5, 6] or the classic by Lamb [33] show that the mathematics is not for the faint of heart.

Examples before theory! Our examples – which illustrate two extremes of fluid flow, oozing and turbulent – are a marble falling through vegetable oil (Section 4.4) and a raindrop falling from the sky after condensing out of a cloud (Section 4.7). In each case we calculate the terminal velocity: the velocity that the object reaches after falling for a sufficiently long time.

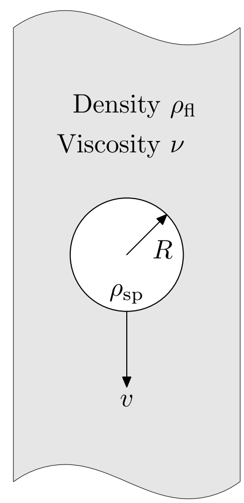

Figure 4.1. *Sphere falling in a liquid. Its terminal velocity is v.*

To find the terminal velocity, solve the partial-differential equations of fluid mechanics for the incompressible flow of a Newtonian fluid:

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} &= -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{v}, \qquad \text{(3 eqns)} \\ \boldsymbol{\nabla}\cdot\mathbf{v} &= 0. \qquad \text{(1 eqn)} \end{aligned} \tag{4.1}$$

Here $\mathbf{v}$ is the fluid velocity, ρ is the fluid density, ν is the kinematic viscosity, and p is the pressure. The first equation is a vector shorthand for three equations, so (4.1) contains four equations. All are partial-differential equations and three are nonlinear. Worse, they are coupled: Quantities appear in more than one equation. So we have to solve a system of coupled, nonlinear, partial-differential equations. This solution must satisfy boundary conditions imposed by the marble or raindrop. As the object moves, the boundary conditions change. So until you know how the object moves, you do not know the boundary conditions. Until you know the boundary conditions, you cannot find the motion of the fluid or of the object. This coupling between the boundary conditions and solution compounds the difficulty of the problem. It requires that you solve the equations and the boundary conditions together. Only then, if you ever get there, can you take the limit $t \to \infty$ to find the terminal velocity.

Sleep easy! We discussed the Navier–Stokes equations (4.1) only to scare you into using dimensional analysis, to convince you that it might be easier than solving partial-differential equations.

4.1 Naive dimensional analysis

To use dimensional analysis, follow the usual steps: Choose relevant variables, form dimensionless groups from them, and solve for the velocity. In choosing variables do not forget to include the variable for which you are solving, which here is v! To decide on the other variables, divide them into three categories (divide and conquer): characteristics of the fluid, of the object, and of whatever makes the object fall. This last category is the easiest, so deal with it first (the principle of maximal laziness). Gravity makes the object fall, so g is on the list.

Consider next the characteristics of the object. Its velocity, as the quantity for which we are solving, is already on the list. Its mass m affects the terminal velocity: A feather falls more slowly than a rock does. Its radius r probably affects the terminal velocity. Instead of listing r and m together, we remix them and use r and ρ_{sp}. The two alternatives provide the same information (assuming the object is uniform): You can compute ρ_{sp} from m and r or compute m from ρ_{sp} and r. You can choose the preferable pair by looking ahead in the derivation. The relevant properties of the fluid will include its density ρ_{fl}. If we use ρ_{sp} then the results might contain ratios such as $\rho_{\text{sp}}/\rho_{\text{fl}}$ (a dimensionless group!). The ratio $\rho_{\text{sp}}/\rho_{\text{fl}}$ has a more obvious physical interpretation than a combination such as $m/\rho_{\text{fl}}r^3$, which, except for a dimensionless constant, is more obscurely the ratio of object and fluid densities. So prefer ρ_{sp} and r over m and r.

Scaling arguments also favor ρ_{sp} and r. In a scaling argument you often imagine varying a size. Size, like heat, is an extensive quantity: a quantity related to amount of stuff. When you vary the size, you want as few other variables as possible to change so that such changes do not obscure the effect of changing size. Whenever you can, replace extensive quantities with **intensive quantities** like temperature or density. The pair m and r contains two extensive quantities, whereas the preferable pair ρ_{sp} and r contains only one.

Now consider properties of the fluid. Its density ρ_{fl} affects the terminal velocity. Perhaps its viscosity is also relevant. Viscosity measures the tendency of a fluid to reduce velocity differences in the flow. You can observe an analog of viscosity in traffic flow on a multilane highway. If one lane moves much faster than another, drivers switch from the slower to the faster lane, eventually slowing down the faster lane. Local decisions of the drivers reduce the velocity gradient. Similarly, molecular motion (in a gas) or collisions (in a fluid) transports speed (really, momentum) from fast- to slow-flowing regions. This transport reduces the velocity difference between the regions. Thicker, oozier (more viscous) fluids probably produce more drag than thin fluids do. So include viscosity in the list.

Fluid mechanicians[1] have defined two viscosities: dynamic viscosity η and kinematic viscosity ν. They are related by $\eta = \rho_{\text{fl}}\nu$. *Life in Moving Fluids* [61, pp. 23–25] discusses the two types of viscosity in detail. In Section 6.6 we estimate the viscosity of gases by examining

1. *Sadly, we could not use the more mellifluous term* fluid mechanics *to signify a host of physicists agonizing over the equations of fluid mechanics; it would not distinguish the toilers from their toil.*

the molecular origin of viscosity. For the analysis of drag force, you need to know only that viscous forces are proportional to viscosity. Which viscosity should we use? Dynamic viscosity hides ρ_{fl} inside the product $\nu\rho_{\text{fl}}$; a ratio of ρ_{sp} and η then looks less dimensionless than it is because ρ_{sp}'s partner ρ_{fl} is buried inside η. Therefore the kinematic viscosity ν usually gives the more insightful results. Summarizing the discussion, Table 4.1 lists the variables by category.

The next step is to find dimensionless groups. The Buckingham Pi theorem (Theorem 3.1) says that the six variables and three independent dimensions result in three dimensionless groups. Before finding them, consider the consequences of having three groups. Three?! The pendulum example at first had one dimensionless group (Section 3.3) with a result of this form for the period τ:

$$\text{group with } \tau = \text{dimensionless constant.} \tag{4.2}$$

Var.	*Dim.*	*Description*
ν	L^2T^{-1}	kinematic viscosity
ρ_{fl}	ML^{-3}	fluid density
r	L	object radius
v	LT^{-1}	terminal velocity
ρ_{sp}	ML^{-3}	object density
g	LT^{-2}	gravity

Table 4.1. *Variables that determine terminal velocity. They are grouped by category: characteristics of the fluid, of the object, and of whatever makes it fall. These six variables contain three independent dimensions.*

Section 3.4 added an extra variable (the release angle), which produced a second group. This complication resulted in a solution of the form:

$$\text{group with } \tau = f(\text{other group}). \tag{4.3}$$

Then we had to deduce the form of f using symmetry and extreme-cases arguments – in other words, by adding physics knowledge. Following the patterns (4.2) and (4.3), three dimensionless groups produce this form for the terminal velocity v:

$$\text{group with } v = f(\text{other group 1}, \text{other group 2}). \tag{4.4}$$

To deduce the properties of f, we would incorporate physics knowledge. However, studying a two-variable function is more onerous than studying a one-variable function. A function of one variable is represented by a curve and can be graphed on a sheet of paper. A function of two variables is represented by a surface. For a complete picture it needs three-dimensional paper (do you have any?); or you can graph many slices of it on regular two-dimensional paper. Neither choice is appealing: The brute-force approach to the terminal velocity produces too many dimensionless groups and results in the difficult (4.4).

If you simplify only after you reach the complicated (4.4), you carry baggage that you eventually discard. When going on holiday to the Caribbean, why pack snow shoes? Instead incorporate physics knowledge at the beginning of the analysis – in other words, now – to simplify the rest of the derivation. To pack light, we'll consider the physics of terminal velocity in order to make simplifications now.

4.2 Simpler approach

The adjective *terminal* in the phrase 'terminal velocity' hints at the physics that determines the velocity. Here 'terminal' is used in its sense of final, after infinite time. It indicates that the velocity has become constant, which happens only when no net force acts on the marble. This line of thought suggests that we imagine the forces acting on the object: gravity, buoyancy, and drag. The terminal velocity is velocity at which the drag, gravitational, and buoyant forces combine to make zero net force. Divide-and-conquer reasoning splits the terminal-velocity problem into three simpler problems (Figure 4.2).

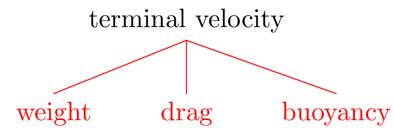

Figure 4.2. *Initial roadmap for finding the terminal velocity. The red items are the ones that we recently discovered. Here we just discovered that terminal velocity can be found from three simpler problems.*

The gravitational force, also known as the weight, is mg. Instead of m we use $(4\pi/3)\rho_{\mathrm{sp}} r^3$ – for the same reasons that we listed ρ_{sp} instead of m in Table 4.1 – and happily ignore the factor of $4\pi/3$. With those choices, the weight is

$$F_{\mathrm{g}} \sim \rho_{\mathrm{sp}} r^3 g. \tag{4.5}$$

Figure 4.3 shows the roadmap updated with this information.

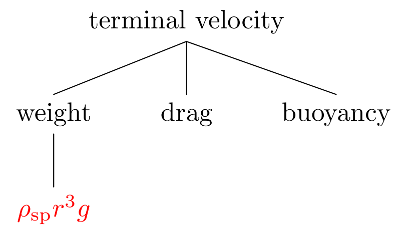

Figure 4.3. *Second roadmap for finding the terminal velocity. We have explored one piece of the landscape laid out in Figure 4.2.*

The remaining pieces are drag and buoyancy. Buoyancy is easier, so do it first (the principle of maximal laziness). It is an upward force that results because gravity affects the pressure in a fluid (Figure 4.4). The pressure increases according to $p = p_0 + \rho_{\mathrm{fl}} gh$, where h is the depth and p_0 is the pressure at zero depth (which can be taken to be at any level in the fluid). The pressure difference between the top and bottom of the object, which are separated by a distance $\sim r$, is $\Delta p \sim \rho_{\mathrm{fl}} gr$. Pressure is force per area, and the pressure difference acts over an area $A \sim r^2$. Therefore the buoyant force created by the pressure difference is

$$F_{\mathrm{b}} \sim A\Delta p \sim \rho_{\mathrm{fl}} r^3 g. \tag{4.6}$$

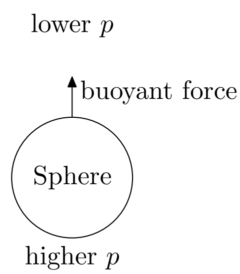

Figure 4.4. *Gravity causes buoyancy. Because of gravity, the pressure at the bottom of the sphere (submerged in a fluid that is not shown) is greater than the pressure at the top. The result is an upward force: buoyancy.*

As a check on this result, Archimedes's principle says that the buoyant force is the 'weight of fluid displaced'. This weight is

$$\overbrace{\rho_{\mathrm{fl}} \underbrace{\frac{4\pi}{3}\pi r^3}_{\textit{volume}}}^{\textit{mass}} g. \tag{4.7}$$

Except for the factor of $4\pi/3$, it matches the buoyant force (4.6), so Archimedes's principle confirms our estimate for F_{b}. That result updates the roadmap to Figure 4.5. The main unexplored branch is the drag force, which we approach using dimensional analysis.

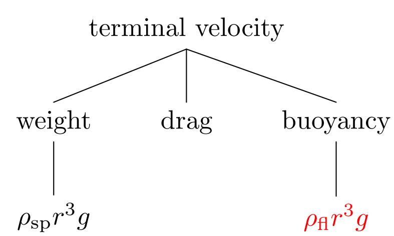

Figure 4.5. *Third roadmap for finding the terminal velocity. We have explored two pieces of the landscape laid out in Figure 4.2.*

4.3 Dimensional analysis for the drag force

The purpose of breaking the problem into parts was to simplify the naive dimensional analysis in Section 4.1. Let's see how the list of variables in Table 4.1 changes now that we are computing the drag force instead of the terminal velocity. The drag force F_{d} joins the list: not a promising beginning to a journey of variable reduction. Worse, the terminal velocity v remains on the list, even though we are no longer computing it, because the drag force depends on the velocity of the object.

However, the drag force has no idea what is inside the sphere. Picture the fluid as a huge computer that implements the laws of fluid dynamics. To this computer, the parameters v and r are the only relevant attributes of a moving sphere. What lies underneath the surface does not affect the fluid flow.[2] The computer can determine the flow (if it has tremendous processing power) without knowing the sphere's density ρ_{sp}, which means it vanishes from the list. Progress! Now consider the characteristics of the fluid. The fluid supercomputer still needs the density and viscosity of the fluid to determine how the pieces of fluid move in response to the object's motion. So ρ_{fl} and ν remain on the list. What about gravity? It causes the object to fall, so it is responsible for the terminal velocity v. However, the fluid supercomputer does not care how the object acquired this velocity; it cares only what the velocity is. So g vanishes from the list. Table 4.2 shows the new, shorter list.

2. *Drag is only skin deep.*

Var.	*Dim.*	*Description*
ν	$\mathrm{L^2T^{-1}}$	kinematic viscosity
ρ_{fl}	$\mathrm{ML^{-3}}$	fluid density
r	L	object radius
v	$\mathrm{LT^{-1}}$	terminal velocity
~~ρ_{sp}~~	~~$\mathrm{ML^{-3}}$~~	~~object density~~
~~g~~	~~$\mathrm{LT^{-2}}$~~	~~gravity~~
F_{d}	$\mathrm{MLT^{-2}}$	drag force

Table 4.2. *Variables that determine the drag force. This list is one variable shorter than the list in Table 4.1.* F_{d} *(in red) joins the party but* g *and* ρ_{sp} *leave.*

The five variables in the list are composed of three basic dimensions. From the Buckingham Pi theorem (Theorem 3.1), we expect two dimensionless groups (Figure 4.6). We find one group by dividing and conquering. The list already includes a velocity (the terminal velocity). If we can concoct another quantity V with dimensions of velocity, then v/V is a dimensionless group. The viscosity ν is almost a velocity. It contains one more power of length than velocity does. Dividing by r eliminates the extra length: $V \equiv \nu/r$. A dimensionless group is then

$$\Pi_1 \equiv \frac{v}{V} = \frac{vr}{\nu}. \tag{4.8}$$

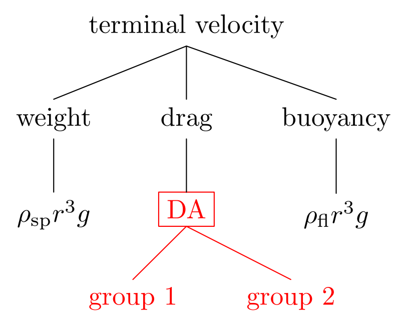

Figure 4.6. *Fourth roadmap for finding the terminal velocity. We are exploring the dimensional-analysis (DA) portion of the landscape, wherein we must discover two dimensionless groups.*

Our knowledge, including this group, is shown in Figure 4.7. This group is so important that it has a name, the **Reynolds number**, which is abbreviated Re. It is important because it is a *dimensionless* measure of flow speed. The velocity, because it contains dimensions, cannot distinguish fast from slow flows (see the discussion of large cargos in Section 1.1): $1000\,\mathrm{m\,s^{-1}}$ is slow for a planet, whose speeds are typically tens of kilometers per second, but fast for a pedestrian. When you hear that a quantity is small, fast, large, expensive, or almost any adjective, your first reaction should be to ask, 'compared to what?' Such a comparison suggests dividing v by another velocity, for then we get a dimensionless quantity that is proportional to v. The result of this division (4.8) is the Reynolds number. Low values of Re indicate slow, viscous flow (cold honey oozing out of a jar); high values indicate turbulent flow (a jet flying at 600 mph). The excellent *Life in Moving Fluids* [61] discusses many dimensionless ratios that arise in fluid mechanics. We discuss two of them: the Froude number for speedboating (Section 8.5) and the Prandtl number for thermal

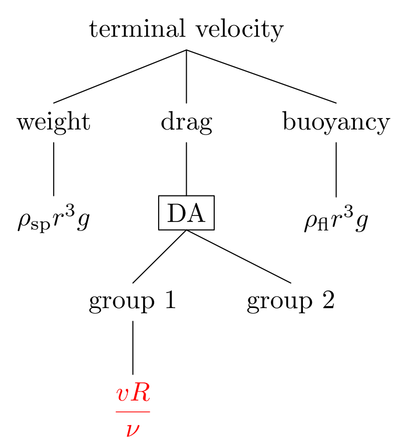

Figure 4.7. *Fifth roadmap for finding the terminal velocity. We just found a first dimensionless group, the Reynolds number.*

boundary layers (Section 10.7).

The Reynolds number looks lonely in Figure 4.7. To bring it company we find a second dimensionless group. The drag force is absent from the first group so it must live in the second; otherwise we cannot solve for the drag force. Instead of dreaming up the dimensionless group in one lucky guess, we construct it in steps (divide-and-conquer reasoning). Examine the variables in Table 4.1 dimension by dimension. Only two (F_d and ρ_{fl}) contain mass, so either both appear in the group or neither appears. Because F_d has to appear, so must ρ_{fl}. Because each variable contains a first power of mass, the group has to contain the ratio F_d/ρ_{fl}. A simple choice is

$$\Pi_2 \propto \frac{F_d}{\rho_{fl}}. \tag{4.9}$$

The dimensions of F_d/ρ_{fl} are L^4T^{-2}, which is the square of L^2T^{-1}. Fortune smiles on us, for L^2T^{-1} are the dimensions of ν. So

$$\frac{F_d}{\rho_{fl}\nu^2} \tag{4.10}$$

is a dimensionless group. This choice, although valid, has a defect: It contains ν, which already belongs to the first group (the Reynolds number). Of all the variables in the problem, ν is the one most likely to be found irrelevant based on a physical argument (as will happen in Section 4.7 when we specialize to high-speed flow). If ν appears in two groups, then eliminating it requires recombining the two groups into one that does not contain ν. However, if ν appears in only one group, then eliminating it is simple: eliminate that group. Simpler mathematics – eliminating a group rather than remixing two groups to get one group – produces clearer thinking, so isolate ν in one group if possible.

To remove ν from the proposed group (4.10), notice that the product of two dimensionless groups is also dimensionless. The first group contains ν^{-1} and the proposed group contains ν^{-2}, so

$$\frac{\text{group proposed in (4.10)}}{(\text{first group})^2} = \frac{F_d}{\rho_{fl} r^2 v^2} \tag{4.11}$$

is also dimensionless. Our knowledge including this group is shown in Figure 4.8.

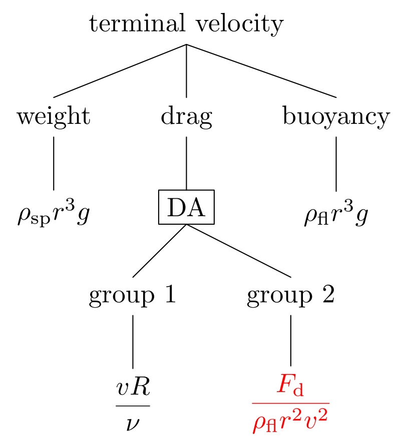

Figure 4.8. *Sixth roadmap for finding the terminal velocity. We just found a second (and final) dimensionless group. This group contains the drag force F_d. We learn its name only in Section 4.7 where it will turn out, except for constant factors, to be the drag coefficient.*

This group, unlike the the proposal (4.10), has a plausible physical interpretation. Imagine that the sphere travels a distance l, and use l to multiply the group by unity:

$$\underbrace{\frac{F_d}{\rho_{fl} r^2 v^2}}_{\textit{group 1}} \times \underbrace{\frac{l}{l}}_{1} = \frac{F_d l}{\rho_{fl} l r^2 v^2}. \tag{4.12}$$

The numerator is the work done against the drag force over the distance l. The denominator is also an energy. To interpret it, examine its parts (divide and conquer). The product lr^2 is, except for a dimensionless constant, the volume of fluid swept out by the object. So $\rho_{\text{fl}}lr^2$ is, against except for a constant, the mass of fluid shoved aside by the object. The object shoves fluid with a velocity comparable to v, so it imparts to the fluid a kinetic energy

$$E_{\text{K}} \sim \rho_{\text{fl}} l r^2 v^2. \tag{4.13}$$

Thus the ratio (4.12) – and hence the group (4.11) – has this interpretation:

$$\frac{\text{work done against drag}}{\text{kinetic energy imparted to the fluid}}. \tag{4.14}$$

In highly dissipative flows, when a lot of energy is burned up by viscosity, the numerator is much larger than the denominator, so this ratio (which will turn out to measure drag) is much greater than 1. In highly streamlined flows (a jet wing), the the work done against drag is small because the fluid returns to the object most of the imparted kinetic energy; the ratio (4.14).

To solve for F_{d}, which is contained in Π_2, use the form $\Pi_2 = f(\Pi_1)$:

$$\frac{F_{\text{d}}}{\rho_{\text{fl}} r^2 v^2} = f\left(\frac{vr}{\nu}\right). \tag{4.15}$$

The drag force is then

$$F_{\text{d}} = \rho_{\text{fl}} r^2 v^2 \, f\left(\frac{vr}{\nu}\right). \tag{4.16}$$

The function f is a dimensionless function: Its argument is dimensionless and it returns a dimensionless number. It is also a universal function, like the dimensionless constant in the area of a circle (3.4) or the function in the area of an ellipse (3.24). So the same f applies to spheres of any size, in a fluid of any viscosity or density! Although f depends on r, ρ_{fl}, ν, and v, it depends on them only through one combination, the Reynolds number. A function of one variable is easier to study than is a function of four variables.[3]

Wind tunnels depend on the universality of f not only to avoid measuring drag at every combination of size and speed but also to make them affordable. Once an engineer designs a new train, it requires extensive testing. Without universality, a full-size model would have to be tested in air blowing at the train's actual speed. Thanks to universality, the engineer can test the train in cheaper conditions with the same Reynolds number: for example, with a scaled-down model moving in water instead of air. Because $\nu_{\text{air}}/\nu_{\text{water}} \sim 15$ and ν appears in the denominator of the Reynolds number, the train in water can be reduced by a factor of 15 in all lengths while preserving the

3. *'A good table of functions of one variable may require a page; that of a function of two variables a volume; that of a function of three variables a bookcase; and that of a function of four variables a library.'*
—Harold Jeffreys [45, p. 82]

Reynolds number (if the speed is unchanged). So instead of building a 500 m × 4 m × 3 m tunnel to fit a full-size train, the engineers can build a tunnel of rough dimensions 30 m × 0.25 m × 0.2 m.

Now we are stuck: Dimensional analysis cannot tell us the form of f, and transparency is not among its many virtues. To learn its form, we specialize to two extremes (the method of extreme cases): viscous, low-speed flow ($\mathrm{Re} \ll 1$) and turbulent, high-speed flow ($\mathrm{Re} \gg 1$).

4.4 Viscous limit

As an example of the low-speed limit, consider a marble falling in vegetable oil. You may wonder how often marbles fall in oil, and why we bother with this example. The short answer to the first question is 'not often'. However, the same physics that determines the fall of marbles in oil also determines, for example, the behavior of fog droplets in air, of bacteria swimming in water [50], or of oil drops in the Millikan oil-drop experiment. The marble problem not only illustrates the physical principles, but also we can check our results with a home experiment.

In slow, viscous flows the drag force comes from – surprise! – viscous forces. These forces are proportional to viscosity. Therefore

$$F_\mathrm{d} \propto \nu. \tag{4.17}$$

The viscosity appears exactly once in the drag (4.16), repeated here:

$$F_\mathrm{d} = \rho_\mathrm{fl} r^2 v^2 \, f\left(\frac{vr}{\nu}\right). \tag{4.18}$$

To flip ν into the numerator and make $F_\mathrm{d} \propto \nu$, the function f must have the form $f(x) \sim 1/x$. [We used $\sim$ to avoid writing a dimensionless constant repeatedly. The $\sim$ symbol means that the two sides have the same dimensions (none in this case) but may differ by a dimensionless constant.] With this $f(x)$ the drag (4.18) becomes

$$F_\mathrm{d} \sim \rho_\mathrm{fl} r^2 v^2 \, \frac{\nu}{vr} = \rho_\mathrm{fl} \nu v. \tag{4.19}$$

Using only dimensional analysis, we cannot compute the missing magic dimensionless constant. A fluid mechanic must do the messy and difficult calculation; we hope that her burden is light now that we have worked out the solution except for this one number. The English mathematician Stokes, the first to derive its value, found that

$$F_\mathrm{d} = 6\pi \rho_\mathrm{fl} \nu v r. \tag{4.20}$$

Let's sanity check this result. Large or fast marbles should feel a lot of drag, so r and v should be in the numerator. Viscous fluids should produce a lot of drag, so ν should be the numerator. The drag force

(4.20) passes these tests. The location of the density (numerator or denominator) is hard to judge. You can make an educated judgment by studying the Navier–Stokes equations (4.1). In those equations, when v is 'small' (small compared to what?) then the $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ term, which contains two powers of v, becomes tiny compared to the viscous term $\nu\boldsymbol{\nabla}^2\mathbf{v}$, which contains only one power of v. The second-order term arises from the inertia of the fluid, so this term's being small says that the oozing marble does not experience inertial effects. So perhaps ρ_{fl}, which represents the inertia of the fluid, should not appear in the Stokes drag (4.20). On the other hand, viscous forces are proportional to the *dynamic* viscosity $\eta = \rho_{\text{fl}}\nu$, so ρ_{fl} should appear even if inertia is unimportant. The drag (4.20) passes this test. Using the dynamic instead of kinematic viscosity, the Stokes drag (4.20) is

$$F_{\text{d}} = 6\pi\eta v r, \tag{4.21}$$

often a convenient form because many tables list η rather than ν.

This factor of 6π in (4.20) comes from doing honest physics: here, from solving the Navier–Stokes equations (4.1). In this book we wish to teach you how not to suffer, so we do not solve such equations. We occasionally quote the factor that honest physics produces, to show you how accurate (or sloppy) the order-of-magnitude approximations are. The honest-physics factor is often near 1, although not in this case where it is roughly 20! In fancy talk, it is usually 'of order unity'. Such a number suits our neural hardware: It is easy to remember and to use. Knowing the order-of-magnitude derivation and remembering this one number, you can reconstruct the exact result without solving difficult equations.

4.5 Terminal velocity for low Reynolds number

Having assembled all the pieces in Figure 4.8, we now return to the original problem of finding the terminal velocity. Since no net force acts on the marble (the definition of terminal velocity), the drag force (4.16) plus the buoyant force (4.6) equals the weight (4.5):

$$\underbrace{\nu\rho_{\text{fl}}vr}_{F_{\text{d}}} + \underbrace{\rho_{\text{fl}}gr^3}_{F_{\text{b}}} \sim \underbrace{\rho_{\text{sp}}gr^3}_{F_{\text{g}}}. \tag{4.22}$$

After rearranging:

$$\nu\rho_{\text{fl}}vr \sim (\rho_{\text{sp}} - \rho_{\text{fl}})gr^3. \tag{4.23}$$

The terminal velocity is then

$$v \sim \frac{gr^2}{\nu}\left(\frac{\rho_{\text{sp}}}{\rho_{\text{fl}}} - 1\right). \tag{4.24}$$

In terms of the dynamic viscosity η, it is

$$v \sim \frac{gr^2}{\eta}(\rho_{\text{sp}} - \rho_{\text{fl}}). \tag{4.25}$$

This version, instead of having the dimensionless factor $\rho_{\text{sp}}/\rho_{\text{fl}} - 1$ that appears in the version (4.24) with kinematic viscosity, has a dimensionful $\rho_{\text{sp}} - \rho_{\text{fl}}$ factor. In that comparison it loses aesthetic points. However, it is often more convenient because tables often list dynamic viscosity (η) rather than kinematic viscosity (ν).

We can increase our confidence in this expression by checking whether the correct variables are upstairs (a picturesque way to say 'in the numerator') and downstairs (in the denominator). Denser marbles should fall faster than less dense marbles, so ρ_{sp} should live upstairs. Gravity accelerates marbles, so g should live upstairs. Viscosity slows marbles, so ν should live downstairs. The terminal velocity (4.24) passes these tests. We therefore have more confidence in our result, although the tests did not check the location of r or any exponents: For example, should ν appear as ν^2? Who knows, but if viscosity matters, it mostly appears as a square root (as in Chapter 10) or as a first power. To check r, imagine a large marble. It will experience a lot of drag and fall slowly, so r should appear downstairs. However, large marbles are also heavy and fall rapidly, which suggests that r should appear upstairs. Which effect wins is not obvious, although after you have experience with these problems, you can make an educated guess: weight scales as r^3, a rapidly rising function r, whereas drag is probably proportional to a lower power of r. As it does here, weight usually wins such contents, leaving r upstairs. So the terminal velocity (4.24) passes this test as well.

Let's look at the dimensionless ratio in parentheses: $\rho_{\text{sp}}/\rho_{\text{fl}} - 1$. Without buoyancy the -1 disappears, and the terminal velocity would be

$$v \propto g\,\frac{\rho_{\text{sp}}}{\rho_{\text{fl}}}. \tag{4.26}$$

We retain the g in the proportionality for the following reason: The true solution (4.24) returns if we replace g by an effective gravity

$$g' \equiv g\left(1 - \frac{\rho_{\text{fl}}}{\rho_{\text{sp}}}\right). \tag{4.27}$$

So, one way to incorporate the effect of the buoyant force is to solve the problem without buoyancy but with the reduced g (4.27). Let's check this replacement in two limiting cases: $\rho_{\text{fl}} = 0$ and $\rho_{\text{fl}} = \rho_{\text{sp}}$. When $\rho_{\text{sp}} = \rho_{\text{fl}}$ gravity vanishes: People, whose density is close to the density of water, barely float in swimming pools. So g' should be zero. When $\rho_{\text{fl}} = 0$ buoyancy vanishes and gravity retains its full effect. So g' should equal g. The effective gravity (4.27) satisfies both tests. Between these two limits, the effective g should vary linearly with ρ_{fl} because buoyancy and weight superpose linearly in their effect on the object. The effective g passes this test as well. Another test is to imagine $\rho_{\text{fl}} > \rho_{\text{sp}}$. Then (4.27) correctly predicts that g' is negative: helium balloons rise. This subtle buoyancy alternative (4.27) is often useful. If for example you forget to include buoyancy (which happened in the first draft of this chapter), you can correct the results at the end by replacing g as in (4.27).

If we carry forward the constants of proportionality, starting with the magic 6π from the drag force (4.20) and including the $4\pi/3$ that belongs in the weight (4.5), we find

$$v \sim \frac{2}{9}\frac{gr^2}{\nu}\left(\frac{\rho_{\text{sp}}}{\rho_{\text{fl}}} - 1\right). \tag{4.28}$$

4.6 Conductivity of seawater

As an application of Stokes drag (4.20) and a rare example of a realistic situation with low Reynolds numbers, we estimate the electrical conductivity of seawater. Solving this problem is hopeless without breaking it into pieces. Conductivity σ is the reciprocal of resistivity ρ. (Apologies for the convention that overloads the density symbol with yet another meaning.) Resistivity, as its name suggests, is related to resistance R. Why have both ρ and R? Resistance is a useful measure for a particular wire, but not for wires in general because it depends on the diameter and cross-sectional area of the wire. Before examining that relationship, let's finish sketching the solution tree, leaving ρ as depending on R plus geometry. We can find R by placing a voltage V across a block of seawater and measuring the current I; then $R = V/I$.

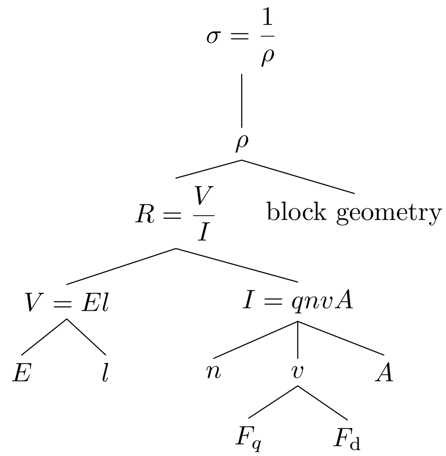

Figure 4.9. Seawater roadmap. The parameters length l and cross-sectional area A are specified in the thought experiment as the geometry of the block; they should cancel in σ, which is constructed to be independent of geometry. The voltage V is also specified as part of the thought experiment, and it should cancel because resistances are independent of voltage (a restatement of Ohm's law).

To find V or I we need a physical model. First, why does seawater conduct at all? Conduction requires the transport of charge, which is produced by an electric field. Seawater is mostly water and table salt (NaCl). The ions that arise from dissolving salt can transport charge. The current is

$$I = qnvA, \tag{4.29}$$

where A is the cross-sectional area of the block, q is the ion charge, n is the ion concentration, and v is its terminal speed. This terminal speed depends on the applied force F_q and on the drag force F_{d}, just as for the falling marble but with an electrical force instead of a gravitational force. The result of this subdividing is the map in Figure 4.9.

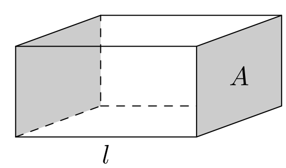

Figure 4.10. Relation between ρ and R. Imagine passing a current between the (shaded) ends of a block of seawater. Doubling the length of the block, which is equivalent to placing two blocks in series, doubles the resistance: $R \propto l$. Doubling the cross-sectional area, which is equivalent to placing two blocks in parallel, halves the resistance: $R \propto A^{-1}$. So $R \propto l/A$. Resistivity ρ is the constant of proportionality. Because that constant does not depend on l or A, it is more interesting than R.

Now let's find expressions for the unknown nodes. Only three remain: ρ, v, and n. Figure 4.10 illustrates the relation between ρ and R:

$$\rho = \frac{RA}{l}. \tag{4.30}$$

To find v we follow the same procedure as for the marble. The applied force is $F_q = qE$, where q is the ion charge. The electric field produced by the voltage V is $E = V/l$, where l is the length of the block, so

$$F_q = \frac{qV}{l}, \tag{4.31}$$

an expression in terms only of known quantities. The drag is Stokes drag (4.21). Equating this drag to the applied force (4.31) gives the terminal velocity v in terms of known quantities:

$$v \sim \frac{qV}{6\pi\eta l r}, \tag{4.32}$$

where r is the radius of the ion. We used the 'roughly equal' symbol $\sim$ rather than an equality, because the physical model of Stokes drag is dubious when the objects are the size of the atoms; at that scale, the fluid hardly looks like a continuum.

Only n remains unknown. We estimate it after getting a symbolic result for σ, which you can do by climbing up the tree in Figure 4.9. First find the current using the terminal velocity from (4.32):

$$I = qnvA \sim \frac{q^2 nAV}{6\pi\eta l r}. \tag{4.33}$$

Use the current to find the resistance:

$$R \sim \frac{V}{I} \sim \frac{6\pi\eta l r}{q^2 nA}. \tag{4.34}$$

The voltage V has vanished, which is encouraging: In most circuits the conductivity (and resistance) is independent of voltage. Use the resistance to find the resistivity:

$$\rho = R\frac{A}{l} \sim \frac{6\pi\eta r}{q^2 n}. \tag{4.35}$$

The expression simplifies as we rise up the tree: The geometric parameters l and A have also vanished, which is also encouraging: The purpose of evaluating resistivity rather than resistance is that resistivity is independent of geometry. Use resistivity to find conductivity:

$$\sigma = \frac{1}{\rho} \sim \frac{q^2 n}{6\pi\eta r}.$$

Here q is the unit charge e or its negative, depending on whether a sodium or a chloride ion is the charge carrier, so

$$\sigma = \frac{1}{\rho} \sim \frac{e^2 n}{6\pi\eta r}. \tag{4.36}$$

To find σ still requires the ion concentration n, which we can find from the concentration of salt in seawater. This value we estimate with a kitchen-sink experiment: Add table salt to a glass of water until it tastes as salty as seawater. We just tried it. In a glass of water a teaspoon of salt is quite salty. A glass of water may have a volume of $0.3\,\ell$ or a mass of $300\,\mathrm{g}$. A flat teaspoon of salt has a volume of about

5 mℓ. For those who live in metric countries, a teaspoon is an archaic measure used in Britain and especially the United States, which has no nearby metric country to which it pays attention. A teaspoon is about 4 cm long by 2 cm wide by 1 cm thick at its deepest point; let's assume 0.5 cm on average. Its volume is therefore

$$\text{teaspoon} \sim 4\,\text{cm} \times 2\,\text{cm} \times 0.5\,\text{cm} \sim 4\,\text{cm}^3. \tag{4.37}$$

The density of salt is maybe twice the density of water, so a flat teaspoon has a mass of $\sim 10\,\text{g}$. The mass fraction of salt in seawater is, in this experiment, roughly 1/30. The true value is remarkably close: 0.035. A mole of salt, which provides two charges per NaCl 'molecule', has a mass of 60 g, so

$$\begin{aligned} n &\sim \frac{1}{30} \times \underbrace{1\,\text{g}\,\text{cm}^{-3}}_{\rho_{\text{water}}} \times \frac{2\,\text{charges}}{\text{molecule}} \times \frac{6\cdot 10^{23}\,\text{molecules}\,\text{mol}^{-1}}{60\,\text{g}\,\text{mol}^{-1}} \\ &\sim 7\cdot 10^{20}\,\text{charges}\,\text{cm}^{-3}. \end{aligned} \tag{4.38}$$

With n evaluated, the only remaining mysteries in the conductivity (4.36) are the ion radius r and the dynamic viscosity η. Do the easy one first. The dynamic viscosity is

$$\eta = \rho_{\text{water}}\nu \sim 10^3\,\text{kg}\,\text{m}^{-3} \times 10^{-6}\,\text{m}^2\,\text{s}^{-1} = 10^{-3}\,\text{kg}\,\text{m}^{-1}\,\text{s}^{-1}. \tag{4.39}$$

We switched to SI (mks) units. Although most calculations are easier in cgs (God's) units than in SI units, the one exception is electromagnetism, which shows up in the e^2 in the conductivity (4.36). Electromagnetism is conceptually easier in cgs units (no ghastly μ_0 or $4\pi\epsilon_0$, for example) than it is in SI units. However, the cgs unit of charge (esu) is so unfamiliar that for numerical calculations we use SI units.

The last quantity required is the ion radius. Each ion is charged so it drags along a shell of water molecules, so rather than use the bare ion radius you should use an augmented radius that includes this shell. But how thick is the shell? As an educated guess, assume that the shell has one layer of water molecules, each with a radius of 1.5 Å. So perhaps $r \sim 2$ Å (Figure 4.11).

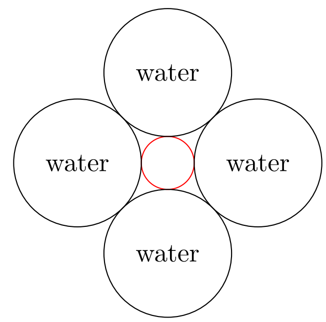

Figure 4.11. *An ion dragging a shell of water molecules. The ion (in red) is charged, and water molecules are polar. So water molecules cluster around the ion.*

With these numbers, the conductivity (4.36) becomes:

$$\sigma \sim \frac{\overbrace{(1.6\cdot 10^{-19}\,\text{C})^2}^{e^2} \times \overbrace{7\cdot 10^{26}\,\text{m}^{-3}}^{n}}{\underbrace{6\times 3}_{6\pi} \times \underbrace{10^{-3}\,\text{kg}\,\text{m}^{-1}\,\text{s}^{-1}}_{\eta} \times \underbrace{2\cdot 10^{-10}\,\text{m}}_{r}}. \tag{4.40}$$

You can do the computation mentally:

> Simultaneously *take out the big part,* apply the *principle of maximal laziness*, and *divide and conquer* by first counting the powers of ten (shown in red) and then worrying about the small factors. Further *divide and conquer* by counting the top and bottom contributions separately. The top contributes -12 powers of ten: -38 from e^2 and $+26$ from n. The bottom contributes -13 powers of ten: -3 from η and -10 from r. Dividing produces one power of ten.
>
> Now do the remaining small factors:
>
> $$\frac{1.6^2 \times 7}{6 \times 3 \times 2}. \tag{4.41}$$
>
> Slightly overestimate the answer by pretending that the 1.6^2 on top cancels the 3 on the bottom. Slightly underestimate the answer – and maybe compensate for the overestimate – by pretending that the 7 on top cancels the 6 on the bottom. After these lies only 1/2 remains. Multiplying it by the sole power of ten gives
>
> $$\sigma \sim 5\,\Omega^{-1}\,\mathrm{m}^{-1}. \tag{4.42}$$
>
> Using a calculator to do the arithmetic gives $4.977\ldots\ \Omega^{-1}\,\mathrm{m}^{-1}$!

The resistivity is

$$\rho \sim \sigma^{-1} \sim 0.2\,\Omega\,\mathrm{m} = 20\,\Omega\,\mathrm{cm}, \tag{4.43}$$

where we converted to the conventional although not fully SI units of $\Omega\,\mathrm{cm}$. A typical value for seawater is $25\,\Omega\,\mathrm{cm}$, which is absurdly close to this estimate!

Probably the most significant error is the radius of the ion-plus-water combination that is doing the charge transport. Perhaps r should be greater than $2\,\text{Å}$, especially for a sodium ion, which is smaller than chloride; it therefore has a higher electric field at its surface and grabs water molecules more strongly than chloride does. In spite of such uncertainties, the continuum approximation produced more accurate results than it ought to. At the length scale of a sodium ion, water looks like a collection of spongy boulders more than it looks like a continuum. Yet Stokes drag worked. It works because the important length scale is not the size of water molecules, but rather their mean free path between collisions. Molecules in a liquid are packed to the point of contact, so the mean free path is much shorter than a molecular (or even ionic) radius, especially compared to an ion with its shell of water. The moral is to approximate first and worry later: Maybe the approximations are correct for reasons that you do not suspect when you start solving a problem. If you agonize over every approximation, you might never begin, and then you will not find out that many approximations would have been fine. . .if only you had had the courage to make them.

4.7 Turbulent limit

We now compute drag in the other flow extreme: high-speed, or turbulent, flow. These results apply to most flows. For example, when a child rises from a chair, the airflow around her is high-speed flow, as you can check by computing the Reynolds number. Say that the child is 20 cm wide, and that she rises with velocity $0.5\,\mathrm{m\,s^{-1}}$. Then

$$\mathrm{Re} \sim \frac{vr}{\nu_{\mathrm{air}}} \sim \frac{50\,\mathrm{cm\,s^{-1}} \times 20\,\mathrm{cm}}{0.2\,\mathrm{cm^2\,s^{-1}}} \sim 5000. \qquad (4.44)$$

Here ν_{air} is closer to $0.15\,\mathrm{cm^2\,s^{-1}}$ than to $0.2\,\mathrm{cm^2\,s^{-1}}$, but $0.2\,\mathrm{cm^2\,s^{-1}}$ easily combines with the 20 cm in the numerator and allows us to do the calculation mentally. Using either value for the viscosity, the Reynolds number is much larger than 1 so the flow is turbulent. Larger objects, such as planes, trains, and automobiles, create turbulence even when they travel even more slowly than the child. Most fluid flow around us is turbulent flow.

Var.	*Dim.*	*Description*
F_{d}	$\mathrm{MLT^{-2}}$	drag force
r	L	object radius
ρ_{fl}	$\mathrm{ML^{-3}}$	fluid density
v	$\mathrm{LT^{-1}}$	terminal velocity

Table 4.3. *Variables that determine the turbulent drag force. Compared to the general list in Table 4.2, this list lacks viscosity.*

To begin the analysis, we assume that a raindrop is a sphere. It is a convenient lie that allows us to reuse the general results of Section 4.3 and specialize to high-speed flow. At high speeds (more precisely, at high Reynolds number) the flow is turbulent. Viscosity – which affects only slow flows but does not influence the shearing and whirling of turbulent flows – becomes irrelevant. Let's see how much we can understand about turbulent drag knowing only that turbulent drag is nearly independent of viscosity. Turbulence is perhaps the main unsolved problem in classical physics. However, you can still understand a lot about drag using dimensional analysis plus a bit of physical reasoning; we do not need a full understanding of turbulence. (You can learn more about turbulence in Chapter 11.) The world is messy so do not wait for a full understanding before you analyze or estimate.

In the roadmap in Figure 4.8 (p. 48), the viscosity appears only in group 1. Because turbulent drag is independent of the viscosity, the viscosity disappears from the results and therefore so does group 2. This argument is glib. More precisely, remove ν from the list in Table 4.2 and search for dimensionless groups. The remaining four variables (Table 4.3) result in one dimensionless group: the second group from Figure 4.8.

So the Reynolds number, which was the first group, has disappeared from the analysis. But why is drag at high speeds independent of Reynolds number? Equivalently, why can we remove ν from the list of variables and still get the correct form for the drag force? The answer is not obvious. Our construction of the Reynolds number (4.8) as a ratio of v and V provides a partial answer – after massaging it. A natural length in this problem is r; we can use r to transform v

and V into times:

$$\begin{aligned} \tau_v &\equiv \frac{r}{v}, \\ \tau_V &\equiv \frac{r}{V} \sim \frac{r^2}{\nu}. \end{aligned} \tag{4.45}$$

Note that $\mathrm{Re} \equiv \tau_V/\tau_v$. The quantity τ_v is the time that fluid takes to travel around the sphere (apart from constants). As we discuss in Section 6.6, kinematic viscosity is the diffusion coefficient for momentum, so the time for momentum to diffuse a distance x is

$$\tau \sim \frac{x^2}{\nu}. \tag{4.46}$$

This result depends on the mathematics of random walks and is derived in Section 6.5; you can increase your confidence in it here by checking that it has valid dimensions (that each side is a time).

So τ_V is the time that momentum takes to diffuse around an object of size r, such as the falling sphere in this problem. If $\tau_V \ll \tau_v$ – in which case $\mathrm{Re} \ll 1$ – then momentum diffuses before fluid travels around the sphere. Momentum diffusion equalizes velocities, if it has time, which it does have in this low-Reynolds-number limit. Momentum diffusion therefore prevents flow at the front from being radically different from the flow at the back, and thereby squelches any turbulence. In the other limit, when $\tau_V \gg \tau_v$ or $\mathrm{Re} \gg 1$ – momentum diffusion is outraced by fluid flow, so the fluid is free to shred itself into a turbulent mess. Once the viscosity is low enough to allow turbulence, its value does not affect the drag, which is why we can ignore it for $\mathrm{Re} \gg 1$. Here $\mathrm{Re} \gg 1$ means 'large enough so that turbulence sets in', which happens around $\mathrm{Re} \sim 1000$. A more complete story, which we discuss as part of boundary layers in Chapter 10, slightly corrects this approximation. However, it is close enough for government work (for the purposes of this chapter).

Here the important point is that the viscosity vanishes from the analysis and so does group 1. Once it disappears, the dimensionless group that remains is

$$\Pi_2 = \frac{F_\mathrm{d}}{\rho_\mathrm{fl} r^2 v^2}. \tag{4.47}$$

The new roadmap is Figure 4.12. Because it is the only group, the solution is

$$\Pi_2 = \text{dimensionless constant}, \tag{4.48}$$

or

$$F_\mathrm{d} \sim \rho_\mathrm{fl} r^2 v^2. \tag{4.49}$$

This drag is for a sphere. What about other shapes, which are characterized by more parameters than a sphere is? So that the drag force generalizes to more complex shapes, we express it using the cross-sectional area of the object. Here $A = \pi r^2$, so

$$F_\mathrm{d} \sim \rho_\mathrm{fl} A v^2. \tag{4.50}$$

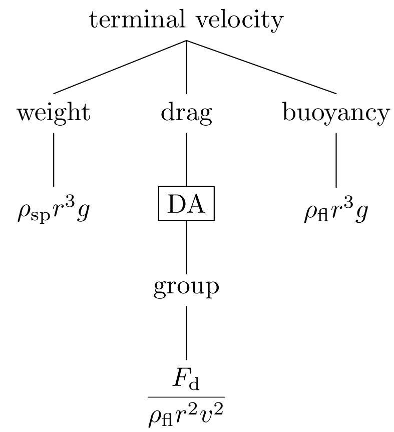

Figure 4.12. *Roadmap for finding the terminal velocity in high-speed flow. After eliminating viscosity from the list of relevant variables, only one dimensionless group remains.*

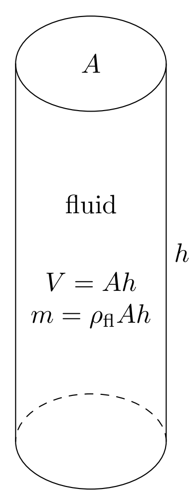

Figure 4.13. *Displaced fluid. An object of cross-sectional area A falls a distance h. It sweeps out a tube of fluid, displacing a mass of $\rho_\mathrm{fl} Ah$.*

This conventional choice has a physical basis. As an object moves, the mass of fluid that it displaces is proportional to its cross-sectional area (Figure 4.13):

$$m_{\text{fl}} = \rho_{\text{fl}} A h. \tag{4.51}$$

The fluid is given a speed comparable to v, so the fluid's kinetic energy is

$$E_{\text{K}} \sim \frac{1}{2} m_{\text{fl}} v^2 \sim \frac{1}{2} \rho_{\text{fl}} A h v^2. \tag{4.52}$$

If all this kinetic energy is dissipated by drag, then the drag force is E_{K}/h or

$$F_{\text{d}} \sim \frac{1}{2} \rho_{\text{fl}} A v^2. \tag{4.53}$$

In this form with the factor of 1/2, rather than in the equivalent (4.50), the constant of proportionality is the **drag coefficient** c_{d}.

Object	c_{d}
Sphere	0.5
Cylinder	1.0
Flat plate	2.0
Car	0.4

Table 4.4. *Approximate drag coefficients (c_{d}) of objects in high-speed flow (Re $\gg 1$). The cylinder moves perpendicular to its axis; the flat plate moves perpendicular to its surface.*

Like the f in the dimensionless drag force (4.15), the drag coefficient is a dimensionless measure of the drag force. It depends on the shape of the object – on how streamlined it is. Table 4.4 lists c_{d} for various shapes. The drag coefficient, being proportional to the function $f(\text{Re})$ in the general solution (4.15), also depends on the Reynolds number. However, using the reasoning that the flow at high Reynolds number is independent of viscosity, the drag coefficient should also be independent of Reynolds number. Using the drag coefficient instead of f (which implies using cross-sectional area instead of r^2), the drag force (4.53) becomes

$$F_{\text{d}} = \frac{1}{2} c_{\text{d}} \rho_{\text{fl}} v^2 A. \tag{4.54}$$

So we have an expression for the turbulent drag force. The weight (4.5) and buoyant forces (4.6) are the same as in the viscous limit. So we redo the analysis of the viscous limit but with the new drag force. Because the weight and buoyant forces contain r^3, we return to using r^2 instead of A in the drag force. With these results, the terminal velocity v is given by

$$\underbrace{\rho_{\text{fl}} r^2 v^2}_{F_{\text{d}}} \sim \underbrace{g(\rho_{\text{sp}} - \rho_{\text{fl}}) r^3}_{F_{\text{g}} - F_{\text{b}}}, \tag{4.55}$$

so

$$v \sim \sqrt{g r \left(\frac{\rho_{\text{sp}}}{\rho_{\text{fl}}} - 1 \right)}. \tag{4.56}$$

Pause to sanity check this result: Are the right variables upstairs and downstairs? We consider each variable in turn.

- ρ_{fl}: The terminal velocity is smaller in a denser fluid (try running in a swimming pool), so ρ_{fl} should be in the denominator.
- g: Imagine a person falling on a planet that has a gravitational force stronger than that of the earth.[4] The drag force does not

4. *Gravity partially determines atmospheric pressure and density. Holding the atmospheric density constant while increasing gravity might be impossible in real life, but we can do it easily in a thought experiment.*

depend on g, so gravity increases the terminal speed without opposition from the drag force: g should be upstairs.

- $\rho_{\rm sp}$: Imagine a raindrop made of (very) heavy water. Relative to a standard raindrop, the gravitational force increases while the drag force remains constant, as shown using the fluid-is-a-computer argument in Section 4.3. So $\rho_{\rm sp}$ should be upstairs.
- r: To determine where the radius lives requires a more subtle argument. Increasing r increases both the gravitational and drag forces. The gravitational force increases as r^3 whereas the drag force increases only as r^2. So, for larger raindrops, their greater weight increases v more than their greater drag decreases v. Therefore r should be live upstairs.
- ν: At high Reynolds number viscosity does not affect drag, at least not in our approximation. So ν should not appear anywhere.

The terminal velocity (4.56) passes all tests.

At last we compute the terminal velocity. The splash spots on the sidewalk made by raindrops in today's rain have $r \sim 0.3\,\mathrm{cm}$. Since rain is water, its density is $\rho_{\rm sp} \sim 1\,\mathrm{g\,cm^{-3}}$. The density of air is $\rho_{\rm fl} \sim 10^{-3}\,\mathrm{g\,cm^{-3}}$, so $\rho_{\rm fl} \ll \rho_{\rm sp}$: Buoyancy is therefore not an important effect, and we can replace $\rho_{\rm sp}/\rho_{\rm fl} - 1$ by $\rho_{\rm sp}/\rho_{\rm fl}$. With this simplification and the estimated numbers, the terminal velocity (4.56) is:

$$v \sim \left(\underbrace{1000\,\mathrm{cm\,s^{-2}}}_{g} \times \underbrace{0.3\,\mathrm{cm}}_{r} \times \frac{\overbrace{1\,\mathrm{g\,cm^{-3}}}^{\rho_{\rm sp}}}{\underbrace{10^{-3}\,\mathrm{g\,cm^{-3}}}_{\rho_{\rm fl}}} \right)^{1/2} \sim 5\,\mathrm{m\,s^{-1}}, \qquad (4.57)$$

or 10 mph.

This calculation assumed that $\mathrm{Re} \gg 1$. Check that assumption! You need not calculate Re from scratch; rather, scale it relative to a previous results. As we worked out in (4.44), a child ($r \sim 20\,\mathrm{cm}$) rising from her chair ($v \sim 0.5\,\mathrm{m\,s^{-1}}$) creates a turbulent flow with $\mathrm{Re} \sim 5000$. The flow created by the raindrop is faster by a factor of 10, but the raindrop is smaller by a factor of roughly 100. Scaling the Reynolds number (4.44) gives

$$\mathrm{Re} \sim \underbrace{\mathrm{Re}_{\rm child}}_{5000} \times \underbrace{\left(\frac{v_{\rm drop}}{v_{\rm child}}\right)}_{10} \times \underbrace{\left(\frac{r_{\rm drop}}{r_{\rm child}}\right)}_{0.01} \sim 500. \qquad (4.58)$$

This Reynolds number is also much larger than 1, so the flow produced by the raindrop is turbulent and our assumption is vindicated.

Now that we have found the terminal velocity, let's extract the pattern of the solution. The order that we followed was *assume*, *derive*, *calculate*, then *check* (Figure 4.14). This order is more fruitful than is the simpler order of *derive* then *calculate*. Without knowing whether the flow is fast or slow, we cannot derive a closed-form expression for F_d; such a derivation is probably beyond present understanding of fluids and turbulence. Blocked by this mathematical Everest, we would remain trapped in the *derive* box. We would never determine F_d, so we would never realize that the Reynolds number is large (the *assume* box); however, only this assumption makes it possible to eliminate ν and thereby to estimate F_d. The moral: Assume early and often!

4.8 Combining solutions from the two limits

You know know the drag force in two extreme cases, viscous (4.20) and turbulent drag (4.54). The results are repeated here:

$$F_d = \begin{cases} 6\pi\rho_{fl}\nu v r & \text{(viscous)}, \\ \frac{1}{2}c_d\rho_{fl}Av^2 & \text{(turbulent)}. \end{cases} \tag{4.59}$$

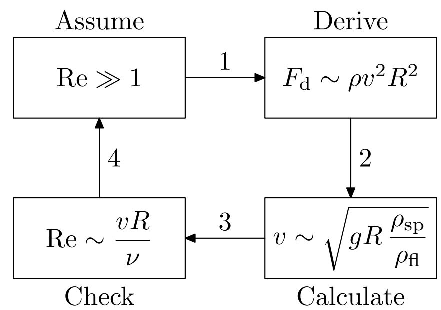

Figure 4.14. *Correct solution order for terminal velocity in turbulent flow. By starting in the* assume *box, we simplified the solution procedure. Step 1: On that assumption, we estimated the drag force (the* derive *box). Step 2: From the drag force, we estimated the terminal velocity (the* calculate *box). Step 3: From the terminal velocity, we estimated the Reynolds number (the* check *box). Step 4: To close the loop, we verified the starting condition.* Note: *For compactness, the terminal-velocity formula ignores the normally small effect of buoyancy.*

Let's compare and combine them by making the viscous form look like the turbulent form. Compared to the turbulent form, the viscous form lacks one power of r and one power of v but has an extra power of ν. A combination of variables with a similar property is the Reynolds number rv/ν. So multiply the viscous drag by a useful form of unity:

$$F_d = \underbrace{\left(\frac{rv/\nu}{\mathrm{Re}}\right)}_{1} \times \underbrace{6\pi\rho_{fl}v\rho_{fl}\nu r}_{F_d} = \frac{1}{\mathrm{Re}}\,6\pi\rho_{fl}v^2r^2 \qquad \text{(viscous)}. \tag{4.60}$$

This form, except for the 6π and the r^2, resembles the turbulent drag in (4.59). Fortunately $A = \pi r^2$ so

$$F_d = \frac{6}{\mathrm{Re}}\,\rho_{fl}v^2A \qquad \text{(viscous)}, \tag{4.61}$$

With

$$c_d = \frac{12}{\mathrm{Re}} \qquad \text{(viscous)}, \tag{4.62}$$

the turbulent drag and this rewritten viscous drag for a sphere have the same form:

$$F_d = \frac{1}{2}\rho_{fl}Av^2 \times \begin{cases} \dfrac{12}{\mathrm{Re}} & (\mathrm{Re} \ll 1), \\ 0.5 & (\mathrm{Re} \gg 1). \end{cases} \tag{4.63}$$

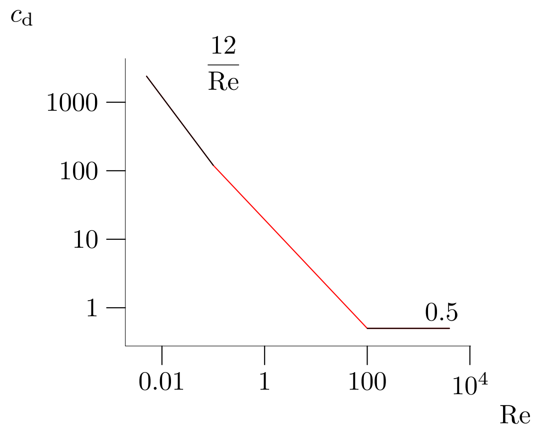

Figure 4.15. *Drag coefficient vs Reynolds number. Reynolds number varies hugely, so a log scale is natural for the x axis. Since cd* $\sim \mathrm{Re}^{-1}$ *for* $\mathrm{Re} \ll 1$*, the drag coefficient also varies widely, and a log scale is natural for the y axis too. The red curve is an an educated guess that interpolates between the two extreme behaviors* $c_d \sim \mathrm{Re}^{-1}$ *and* $c_d \sim 1$*. [Rather, it will be an educated guess once I figure out what's wrong with either the figure-drawing program (*`metapost`*) or with my understanding of it, so that I can convince it to draw splines instead of polygonal paths.]*

At high Reynolds number the drag coefficient remains constant. For a sphere, that constant is $c_d \sim 1/2$. If the low-Reynolds-number approximation for c_d is valid at sufficiently high Reynolds numbers, then c_d would cross 1/2 near $\mathrm{Re} \sim 24$, where presumably the high-Reynolds-number approximation takes over (Figure 4.15). The crossing point is a reasonable estimate for the transition between low- and high-speed flow. Experiment or massive simulation are the only ways to get a more accurate result. Experimental data place the crossover near $\mathrm{Re} \sim 5$, at which point $c_d \sim 2$. Why can't you calculate this value analytically? If a dimensionless variable, such as the Reynolds number, is close to unity, calculations become difficult. Approximations that depend on a quantity being either huge or tiny are no longer valid. When all terms in an equation are roughly of the same magnitude, you cannot get rid of any term without making large errors. To get results in these situations, you have to do honest work: You must do experiments or solve the Navier–Stokes equations (4.1) numerically.

4.9 What you have learned

- *Eliminate early:* To simplify a dimensional analysis, eliminate variables early in the derivation. Do not carry them until the end and then use physics knowledge to eliminate them: Why pack skis when going to the Caribbean?
- *Use intensive quantities:* To facilitate scaling arguments, replace extensive quantities, such as mass, with intensive quantities, such as density.
- *Apply maximal laziness:* Do easy computations before hard computations.
- *Take out the big part:* This maxim is related to 'maximal laziness' because the big part, such as powers of ten in a numerical computation, are usually easier to manipulate than are the little factors.
- *Divide and conquer (again):* Break a problem into parts (divide

and conquer), as with the terminal velocity, to help you simplify a dimensional analysis.

- *Honor the Reynolds number:* The Reynolds number is a dimensionless measure for flow speed. It has several physical interpretations: It distinguishes viscous from turbulent flow, and it compares the momentum transport time to the fluid transport time.
- *Find physical interpretations for dimensionless groups*: Dimensionless groups usually have intuitive physical meanings. For example, the drag coefficient is a ratio of dissipative and kinetic energies.
- *Multiply by unity:* Choose forms of unity to simplify the mathematics or to help make physical interpretations (as we did in interpreting the drag coefficient as a ratio of energies).
- *Search for universal functions:* Dimensionless groups, such as the Reynolds number, and functions such as the f in the drag force (4.16), are universal. Their properties are not specific to a particular solution.
- *Consider extreme cases (again)*: When you solve a difficult problem – such as computing the drag force – simplify by studying extreme cases. Assume that one or more of the dimensionless variables are nearly zero or nearly ∞.
- *Check consistency:* Use your solution to check whether your assumption of an extreme case is valid, as we checked the high-Reynolds-number assumption for the fall of a raindrop.
- *Have courage:* As a corollary of the previous maxim, make assumptions without fear, for you can always check them at the end of a derivation. If you do not assume, you often cannot start.

In the next chapter we use several of these dishonest methods to study the mechanical properties of materials.

4.10 Exercises

▶ **4.18** *Fog*

Fog droplets are typically 10 μm in diameter. Can a low fog, a frequent visitor to the English landscape, remain aloft overnight?

▶ **4.19** *Swimming, cycling*

What is the drag force on a racing cyclist? On an Olympic swimmer? Use these results to estimate how fast people can cycle or swim.

▶ **4.20** *Home experiment*

Test the results for viscous flow with a home experiment. Check that your assumption of viscous flow is consistent!

▶ **4.21** *Golf ball*

What is the Reynolds number of a golf ball after it is hit hard? How does the drag force compare with the weight?

► **4.22** *Raindrop check*
How far does the raindrop fall before it reaches terminal velocity? Sketch its velocity versus the distance fallen. This calculation is a useful check on the analysis in Section 4.7. The distance should be significantly less than the altitude of a cloud. Otherwise the terminal-velocity calculation is relevant to the fall of raindrops.

► **4.23** *Free-fall time, take 2*
Extend the results of the previous exercise to estimate the time for a marble to fall from a height h, including air resistance. **Sketch** (rather than accurately plot) τ versus h and label interesting regions.

► **4.24** *Seawater check*
The derivation in Section 4.6 used Stokes drag, which requires that $\text{Re} \ll 1$. Choose any reasonable current density and estimate the Reynolds number for an ion. Is the derivation consistent?