14 Weather

How much rain falls on the surface of the earth? What determines the average temperature on the earth? Why are there seasons? To answer such questions, this chapter combines techniques of approximation with the results on thermal and mechanical properties of materials.

14.1 Basic numbers

Much of weather depends on only a few physical parameters. To decide which ones, think about the causes of rain and wind. Rain requires that water be lifted into the atmosphere. This process requires energy. Wind, which is the motion of air, requires energy. The energy comes mostly from the sun. An important parameter is therefore the **solar constant**

$$S \sim 1400\,\mathrm{W\,m^{-2}}, \tag{14.1}$$

which is the energy flux (power per area) from the sun *at the top of earth's atmosphere.* As radiation travels into the atmosphere, clouds and water vapor reflect portions of it into space. So the flux reaching the ground is less than S. The difference between the ground and the top-of-atmosphere flux is why we carefully specify where S is measured.

The energy required to move air depends on the density of air,

$$\rho_{\mathrm{air}} \sim 1.3 \cdot 10^{-3}\,\mathrm{g\,cm^{-3}} \tag{14.2}$$

at standard temperature and pressure. Many properties of the atmosphere, such as its density, depend on the mass of air molecules. Air is mostly N_2, so

$$m_{\mathrm{air}} \sim 30 m_{\mathrm{p}}, \tag{14.3}$$

where m_{p} is the mass of a proton.

Water (rain, sleet, snow, ...) is a major component of weather. Its surface tension is, repeating the value quoted in (5.49):

$$\gamma_{\mathrm{water}} \sim 0.07\,\mathrm{N\,m^{-1}}. \tag{14.4}$$

Water's heat of vaporization, which affects rainfall and the properties of clouds, is related to its surface tension and cohesive energy ϵ_c. For convenience, here is its value:

$$L_{\mathrm{vap}}^{\mathrm{water}} \sim 40.7\,\mathrm{kJ\,mol^{-1}} \sim 2.3\,\mathrm{MJ\,kg^{-1}}. \tag{14.5}$$

Water's specific heat defines the calorie:

$$c_{\mathrm{p}}^{\mathrm{water}} = 1\,\mathrm{cal\,g^{-1}\,K^{-1}} \sim 4\,\mathrm{kJ\,kg^{-1}\,K^{-1}}. \tag{14.6}$$

Let's see how much these basic numbers can explain.

14.2 Earth's temperature

How warm does sunlight make the earth's surface? The energy flux is (14.1). How can it become a temperature? Energy and temperature, for one molecule, are related by Boltzmann's constant:

$$\text{energy} \sim k \times \text{temperature}. \tag{14.7}$$

But energy *flux* is energy per area per time. What is the right area and time to convert flux to energy? None is obvious, so instead write $S \propto f(T)$, where T is earth's surface temperature. Now look for a constant of proportionality and the function f. One useful method is **trolling**: to hunt through a table of constants until you find one with the right combination of dimensions.

For example, suppose that you want to know the relation between the temperature of a gas and the average kinetic energy of a molecule of the gas. Start with $E \propto g(T)$, where the constant of proportionality and the function g are unknown. As your eyes tire from the tiny type size in most tables of physical constants, you notice that one constant, k, has units of $\mathrm{J\,K^{-1}}$. So it connects energy and temperature. A likely guess is that $E \sim kT$. In classical physics, this relation is correct except for a dimensionless constant.

The same method turns $S \propto f(T)$ into a useful relation. The constant of proportionality must connect temperature, which is K in SI units, and power per area, which is $\mathrm{W\,m^{-2}}$ in SI units. The Stefan–Boltzmann constant

$$\sigma \sim 6\cdot 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}} \tag{14.8}$$

uses these building blocks. It has flux upstairs and four powers of K downstairs, so the product σT^4 has dimensions of flux. This happy result suggests that

$$S \sim \sigma T^4. \tag{14.9}$$

Before worrying about the physical meaning of this equation, as a sanity check use it to estimate the earth's temperature:

$$\begin{aligned} T_{\mathrm{earth}} &\sim \left(\frac{S}{\sigma}\right)^{1/4} \\ &\sim \left(\frac{1.4\cdot 10^{3}\,\mathrm{W\,m^{-2}}}{6\cdot 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}}\right)^{1/4} \sim 400\,\mathrm{K}. \end{aligned} \tag{14.10}$$

This temperature is high; the oceans would boil dry and turn earth into a hot greenhouse like Venus. However, it is in error by only 30 percent. The guess (14.9) contains a lot of correct physics but is not the whole story.

Before improving it, we investigate the Stefan–Boltzmann constant σ. The temperature (14.10) depends on the looked-up dimensions of the σ. Because σ is imporant enough to have its own Greek letter (a scarce resource!), it could be a new fundamental constant of nature. However, dimensional analysis explains its origin in terms of familiar constants of nature. Radiation is photons, which are relativistic and quantum mechanical, so σ depends on $\hbar$ (quantum mechanics) and c (relativity). It also contains temperature, so Boltzmann's constant k is relevant, as the name 'Stefan–Boltzmann' also suggests. This list (Table 14.1) of four variables and *four* dimensions (length, mass, time, and temperature) produces zero dimensionless groups. Trouble! Before figuring out the problem, simplify it by doing easy steps first (maximal laziness). The temperature dimensions Θ appears in only two variables, so examine it first. Since σ contains Θ^{-4} and k contains Θ^{-1}, the dimensionless group contains σ/k^4. So replace σ and k by σ/k^4 to get a shorter list (Table 14.2). The problem repeats itself: Three variables and three dimensions also produce zero dimensionless groups. However, it repeats itself in a problem with only three dimensions rather than with four dimensions. You could solve this problem, but an alternative approach is to find a dimensionless group anyway, and maybe its form will explain the paradox.

Var.	*Dim.*	*Description*
σ	$\mathrm{MT}^{-3}\Theta^{-4}$	SB constant
$\hbar$	$\mathrm{ML}^2\mathrm{T}^{-1}$	quantum
c	LT^{-1}	relativity
k	$\mathrm{ML}^2\mathrm{T}^{-2}\Theta^{-1}$	Boltzmann

Table 14.1. *Variables that determine the Stefan–Boltzmann constant (in red). Here Θ represents the dimension of temperature.*

Following the usual practice, see whether any dimensions are in only two variables. In this case, it is mass, which is in $\hbar$ (one power) and σ/k^4 (minus three powers). To divide out mass, the group must contain $\hbar^3\sigma/k^4$. This combination has dimensions of $\mathrm{L}^{-2}\mathrm{T}^2$, as does c^{-2}. So a dimensionless group is

$$\Pi_1 = \frac{\sigma\hbar^3c^2}{k^4}. \tag{14.11}$$

Var.	*Dim.*	*Description*
σ/k^4	$\mathrm{M}^{-3}\mathrm{L}^{-8}\mathrm{T}^5$	necessary
$\hbar$	$\mathrm{ML}^2\mathrm{T}^{-1}$	quantum
c	LT^{-1}	relativity

Table 14.2. *Variables that determine σ/k^4 (in red).*

This form explains the apparent lack of a dimensionless group. The dimensions L, M, and T are not independent in this problem. Here, the dimensions of $\hbar$ and c suffice to construct the dimensions of σ/k^4:

$$\left[\frac{\sigma}{k^4}\right] = \left[\hbar^{-3}\right] \times \left[c^{-2}\right], \tag{14.12}$$

where the [bracket] notation indicates 'dimensions of'. So any independent set of dimensions contains only two members, for which reasonable choices are the dimensions of $\hbar$ and c. Three variables ($\hbar$, c, and σ/k^4) and two independent dimensions produce one dimensionless group (14.11). With only one dimensionless group in the problem, the solution is $\Pi_1 =$ constant. Therefore

$$\sigma = \frac{\pi^2}{60}\frac{k^4}{\hbar^3c^2}, \tag{14.13}$$

where the $\pi^2/60$ factor in red is the result of honest calculation. Using $h = 2\pi\hbar$ instead of $\hbar$, the constant becomes $2\pi^5/15$:

$$\sigma = \frac{2\pi^5}{15}\frac{k^4}{h^3c^2}. \tag{14.14}$$

Numerically, $2\pi^5/15 \approx 41$ and $\pi^2/60 \approx 1/6$. As usual, using $\hbar$ instead of h produces dimensionless constants that are closer to unity. This choice makes the order-of-magnitude hope, that 'any unknown dimensionless constant is unity', more accurate. Using $\hbar$, the Bohr radius (5.8) comes out exactly correct (the missing dimensionless constant is unity); with h instead, the dimensionless constant would be $1/4\pi^2 \approx 0.025$. Using angular frequency ω rather than oscillation frequency ν gives a similar benefit. If you use ω, the dispersion relation (8.27) for gravity waves on deep water is exact. To improve the chances of getting a useful result when you approximate, use variables based upon 1 radian rather than upon 1 cycle of oscillation.

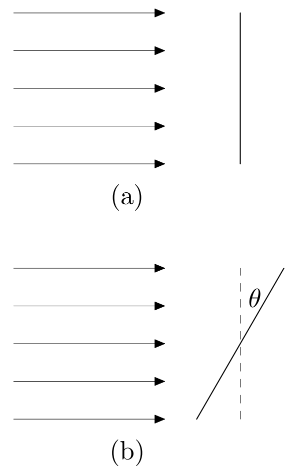

Figure 14.1. *Tilted incidence reduces flux. The same power falls on a segment of earth's surface at perpendicular incidence in (a) as at tilted incidence in (b). However, the segment in (a) is shorter than the segment in (b) by* $\cos\theta$. *So tilted incidence reduces flux by* $\cos\theta$.

14.3 Seasons and nights

The estimate (14.9) for earth's temperature was in error by roughly 30 percent. The error can be reduced by thinking about the geometry of sunlight hitting earth. The estimate came from analyzing dimensions. The correct law of that form is

$$P = \sigma T^4, \tag{14.15}$$

the **Stefan–Boltzmann law**. The terminology is awkward. In it, P is the flux (power per area) radiated from a surface at temperature T. To find the temperature of earth's surface, you need to know P. The estimate of $T_{\text{earth}} \sim 400\,\text{K}$ is based on using, for P, the solar flux at the top of the atmosphere rather than the flux reaching the ground. This substitution ignores factors such as reflection by clouds, but those factors are the superficial problem. A fundamental problem is that it fails at least one-half of the time: At night, the solar flux reaching the ground is zero. Night reduces the effective solar flux by a factor of 2.

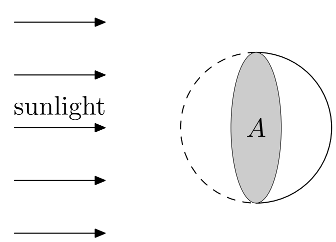

Figure 14.2. *Solar radiation that hits the cross-section of area* πr^2 *is spread over a surface area* $4\pi r^2$, *which is four times larger than the cross-section. So the flux, averaged over the surface, is a factor of 4 smaller than the perpendicular flux.*

A second fundamental problem is that it neglects seasons: Summers are colder than winters because the sunlight hits the ground at an angle. A piece of ground tilted away from the perpendicular intercepts less sunlight than when it faces the sun directly (Figure 14.1). This effect compounds the reduction due to night. The night-day reduction factor is 2, but the incidence factor is not so obvious because the angle of incidence varies over the surface. The calculation requires integrating $\cos\theta$ over the surface of a sphere, where θ is the latitude. An alternative is to invoke a rule of thumb: 'When in doubt, throw in a factor of two.' In emergencies, use this rule or even do an integral. Here you can compute the complete reduction factor including the effect of night using a geometric argument (Figure 14.2). This argument produces a combined reduction factor of 4. This factor can be interpreted in terms of two reduction factors:

$$S_{\text{eff}} = S \times \frac{1}{4} = S \times \underbrace{\frac{1}{2}}_{\textit{night}} \times \underbrace{\frac{1}{2}}_{\textit{tilt}}, \tag{14.16}$$

The rule of thumb was correct: Tilt reduces the average flux by a factor of 2.

The rule of thumb has a basis in an earlier principle: Talk to your gut. Tilt cannot reduce the flux to zero because a large band of surface near the equator faces the sun and receives almost the full flux. Another extreme is the regions near the poles, which intercept little sunlight. If all the surface were at the poles, then S would be multiplied by 0. If all the surface were at the equator (the earth as a one-sided sheet facing the sun), then S would be multiplied by 1. Where between 0 and 1 is the correct latitude factor? A reasonable compromise is 1/2 because the two extremes have equally plausible arguments. The geometry in Figure 14.2 shows that compromise value is the correct value.

The factor of 1/4 is not the final chapter in this story. As mentioned, clouds reflect solar radiation before it hits the surface; so does the atmosphere. A small fraction also is reflected from the surface (polar icecaps, for example) rather than being absorbed and then radiated. The net reflection is composed of:

	Fraction reflected
clouds	0.20
atmosphere	0.06
surface	0.04
total	**0.30**

The total 0.30 is earth's **albedo**, usually labeled a. The remaining fraction, 0.70, is absorbed, contributes to thermal equilibrium, and is radiated as blackbody radiation. So the factor of 1/4 is multiplied by $1 - a = 0.7$. The effective surface flux is then

$$S_{\text{eff}} \approx 0.7 \times \frac{S}{4} \sim \frac{S}{5.5} \sim 250\,\text{W}\,\text{m}^{-2}. \tag{14.17}$$

To find the temperature that this flux produces, you can redo the calculation in (14.10), now with

$$T_{\text{earth}} \sim \left(\frac{S_{\text{eff}}}{\sigma}\right)^{1/4} = \ldots. \tag{14.18}$$

However, do not calculate from scratch. Instead, scale the old result that

$$\text{temperature} \propto (\text{flux})^{1/4}. \tag{14.19}$$

Changing from S to S_{eff} drops the flux by a factor of 5.5, so the temperature T_{earth} drops by $5.5^{1/4} \sim 1.65$. The previous estimate was $T_{\mathrm{earth}} \sim 400\,\mathrm{K}$, so now

$$T_{\mathrm{earth}} \sim \frac{400\,\mathrm{K}}{1.65} \sim 255\,\mathrm{K} \qquad (14.20)$$

or $-18\,^{\circ}\mathrm{C}$. Instead of the oceans' boiling dry, the problem with the estimate of $T_{\mathrm{earth}} \sim 400\,\mathrm{K}$, the oceans freeze solid. This temperature is perhaps $30\,^{\circ}\mathrm{C}$ too low, an error of roughly 10 percent, so it is more accurate than the first estimate.

14.4 How thick is the atmosphere?

A fundamental property of the atmosphere is its thickness. The thickness affects how much solar energy it absorbs, how high clouds can reach, and the color of sunsets. This section presents several methods to estimate the thickness, called H. The atmosphere does not end abruptly at the height H. Rather, it fades gradually into outer space. As usual in order-of-magnitude physics, replace the complicated fade out with a one-layer approximation: Pretend that the atmosphere is uniform, say in density, until the height H, where it ends abruptly (Figure 14.3). In this model, choose any height at which the density or pressure is significantly different from the corresponding value at sea level. The usual choice is the height where the desnity or pressure has fallen by a factor of e. This choice preserves the total mass of the atmosphere in changing from the true curve to the rectangle approximation.

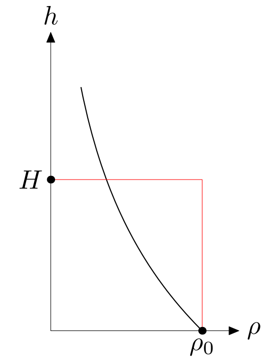

__Figure 14.3.__ Rectangular approximation to the atmosphere. The axes are unusual: height is on the vertical axis, which is intuitive since h measures the vertical direction. In a constant-temperature atmosphere, the density is given by the black curve. The rectangular approximation replaces it by the red rectangle: a constant, sea-level density up to a height H, and zero density above H.

To make a list of relevant variables, consider the physics that determines the height. One source of ideas is thinking about the atmosphere on other heavenly bodies. The moon has almost no atmosphere because the atmosphere escaped eons ago. It escaped because the moon's gravity is weak, and all the molecules had enough thermal speed to escape the moon gravity. On earth only the lightest components of the primordial atmosphere, such as helium, had sufficient speed to escape. So kT is an important factor in the composition of the atmosphere, and perhaps in its thickness. The molecular mass m affects the thermal speed, so it is also relevant. Gravity pulls the atmosphere close to the surface, so g is on the list. As usual, use kT instead of k and T separately; the combination kT avoids introducing a fourth fundamental dimension (temperature). And the list includes H, the goal quantity. Four variables made of three dimensions (Table 14.3) produce one dimensionless group.

Var.	*Dim.*	*Description*
H	L	atmosphere thickness
kT	$\mathrm{ML}^2\mathrm{T}^{-2}$	thermal energy
m	M	molecular mass
g	LT^{-2}	gravity

__Table 14.3.__ Variable that determine the thickness of the atmosphere (in red).

The group is easy to find because kT and mgH are energies, so their ratio is dimensionless:

$$\Pi_1 \equiv \frac{mgH}{kT}. \qquad (14.21)$$

This group has the natural interpretation of

$$\Pi_1 = \frac{\text{gravitational energy}}{\text{thermal energy}}. \tag{14.22}$$

The atmosphere's thickness is a competition between gravity pulling it close to earth and thermal motion pushing it out. Much of the materials physics has a similar structure. Tabor introduces his excellent book *Gases, Liquids and Solids and other states of matter* [68, p. xiii] with this comment:

> The main theme is that the three primary states of matter are the result of competition between thermal energy and intermolecular forces.

Since Π_1 is the only dimensionless group, the general solution is $\Pi_1 \sim 1$ or

$$H \sim \frac{kT}{mg}. \tag{14.23}$$

Now look upstairs and downstairs. High temperature agitates the molecules and increases their chances of escaping, so T should be upstairs. Gravity pulls molecules to the surface, so g and m should be downstairs. The predicted height passes these tests. As a further sanity check, put in numbers. If you multiply by unity, you avoid looking up many constants:

$$H \sim \frac{kT}{mg} = \frac{\overbrace{N_\mathrm{A}k}^{R}T}{(N_\mathrm{A}m)g}, \tag{14.24}$$

where R is the universal gas constant $8.3\,\mathrm{J\,mol^{-1}\,K^{-1}}$. The product $N_\mathrm{A}m$ is $30N_\mathrm{A}m_\mathrm{p}$ since air is mostly diatomic nitrogen with mass given by (14.3). Since $N_\mathrm{A}m_\mathrm{p} = 10^{-3}\,\mathrm{kg}$ by definition, the height H becomes:

$$H = \frac{8.3\,\mathrm{J\,mol^{-1}\,K^{-1}} \times 300\,\mathrm{K}}{3\cdot 10^{-2}\,\mathrm{kg\,mol^{-1}} \times 10\,\mathrm{m\,s^{-2}}} \sim 10\,\mathrm{km}. \tag{14.25}$$

This estimate is reasonable. Climbers ascending Mount Everest ($h \sim 9\,\mathrm{km}$) carry oxygen because air atop Everest contains significantly less oxygen than air at sea level.

This dimensional analysis result (14.23) has many physical interpretations. A natural interpretation considers the atmosphere as a column of air. At sea level, the pressure p_0 holds up a column of air with height H, so $p_0 = \rho_\mathrm{air}gH$ (Figure 14.4) or

$$H \sim \frac{p_0}{\rho_\mathrm{air}g}. \tag{14.26}$$

The values of g and ρ_air are known, so if you can estimate p_0 you can find H. One estimate comes from inflating car tires at a gas station.

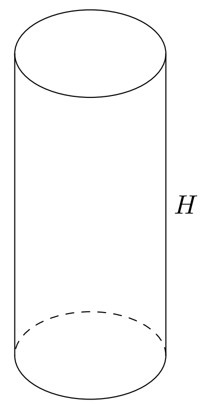

Figure 14.4. *Column of air. Using the constant-density assumption, the pressure at the bottom of the column is $p_0 = \rho_\mathrm{air}gH$.*

The tire-pressure gauges read 'gauge pressure'. A few gauges, at least in the United States, tell you that this pressure is relative to 15 psi ('pounds per square inch'), which is atmospheric pressure:

$$p_0 \sim \frac{15\,\text{lbs force}}{1\,\text{in}^2} \sim \frac{\overbrace{7\,\text{kg}}^{15\,\text{lbs}} \times \overbrace{10\,\text{N}\,\text{kg}^{-1}}^{g}}{(\underbrace{0.025\,\text{m}}_{1\,\text{in}})^2} \sim 10^5\ \text{Pa}. \tag{14.27}$$

So the height is

$$H \sim \frac{10^5\ \text{Pa}}{1\,\text{kg}\,\text{m}^{-3} \times 10\,\text{m}\,\text{s}^{-2}} \sim 10\,\text{km}. \tag{14.28}$$

Another estimate of H comes from the ideal gas law, $p = nkT$, where n is the number of molecules per volume. Because $n = \rho_{\text{air}}/m$, the pressure p is $p = mngH$ so

$$nkT = mngH, \tag{14.29}$$

where the number density n divides out to produce $H = kT/mg$.

As another interpretation of H, imagine launching air molecules into the sky. Air molecules are in constant motion, with an average kinetic energy $E \sim kT$, so a typical launch velocity would be $v \sim \sqrt{kT/m}$. If an air molecule launches towards the sky with that velocity, how high will it reach? This height is, strangely, an estimate of H. It is given by

$$\underbrace{mgH}_{PE} \sim \underbrace{kT}_{KE}, \tag{14.30}$$

so again

$$H \sim \frac{kT}{mg}. \tag{14.31}$$

In an isothermal (constant-temperature) atmosphere, statistical mechanics gives a final estimate of the height. The probability for a molecule of mass m to be at a height h is given by the Boltzmann factor:

$$\text{prob} \propto \exp\left(-\frac{\overbrace{mgh}^{E}}{kT}\right), \tag{14.32}$$

where $E = mgh$ is the potential energy required to bring a molecule to a height h. The density is proportional to the probability, so

$$\rho \propto e^{-mgh/kT}. \tag{14.33}$$

A reasonable definition for H is the height where the density has fallen by a factor of e relative to its value at sea level. Then the dimensionless exponent $-mgH/kT$ is -1 and the height H is again $H \sim kT/mg$. With so many methods confirming the same result, it must be right!