Solutions to Problem Set 1

Physics 103c Spring 1995

1. Pencils
   a) How many atoms thick is the graphite layer left by a pencil writing on a piece of paper? [Depending on your background, you may answer this theoretically, or by performing an experiment, or both]

Performing an experiment I find that drawing 100 lines of 15 cm $\times$ 1/2 mm wears away 1 mm of pencil lead that is 1/2 mm in diameter. The volume of lead worn away is $\sim .2\ \mathrm{mm}^3$ and the area covered is $\sim 7500\ \mathrm{mm}^2$, so that the average thickness of the layer is $t = 0.2/7500 = 3 \times 10^{-5}\,\mathrm{mm} = 3 \times 10^{-6}\,\mathrm{cm}$. Graphite has density of about twice that of water so that graphite has number density approximately

$$n \sim \frac{2\ \mathrm{gm}}{\mathrm{cm}^3} \times \frac{\mathrm{nucleons}}{2 \times 10^{-24}\ \mathrm{gm}} \times \frac{\mathrm{atoms}}{12\ \mathrm{nucleons}} = 8 \times 10^{22}\ \mathrm{atoms\ cm}^{-3}\,.$$

Thus the mean separation between atoms is roughly $a = (8 \times 10^{-22}\ \mathrm{atoms\ cm}^{-3})^{-1/3} = 2 \times 10^{-8}\,\mathrm{cm}$. Hence there are about $t/a \sim 100$ atoms per layer.

Theoretical answer: to look black, the graphite must absorb all colors in the visible spectrum of light, i.e., over the frequency range $\Delta\nu \sim \nu \sim c/\lambda \sim 6 \times 10^{14}\,\mathrm{Hz}$ for green light ($\lambda = 500\,\mathrm{nm}$). The frequency-integrated cross-section for an oscillator coupled to electromagnetic radiation is

$$\int \sigma\, d\nu = \frac{\pi e^2}{m_e c} f\,,$$

where $f = 1$ classically, and is $\mathcal{O}(1)$ for strong quantum mechanical transitions. Thus for graphite, $\sigma \sim (\pi e^2/m_e c \Delta\nu) \sim 4 \times 10^{-17}\mathrm{cm}^{-2}$. To look black, we need an optical depth $n\sigma t$ of several, say 3. Then $t > 3/(n\sigma) \sim 10^{-6}\,\mathrm{cm}$, which is 3 times smaller than the experimental result. Most pencil 'lead' is clay mixed with graphite, and the clay will contribute to t without contributing much to the absorption. This may account for the discrepancy. Or perhaps we are making the lines too thick, or $f \sim 0.3$. Anyone want to figure out which?

   b) How many words can a standard No. 2 pencil write (assume you don't waste lead by over-sharpening)?

A No. 2 pencil has diameter of lead ~ 2 mm and is $l \sim 15$ cm long which gives it a usable volume of lead of $\sim 500\ \mathrm{mm}^3$. However people usually keep pencils sharpened to $d = 1/2$ mm points, so the usable volume of lead is only about $ld^2 = 40\,\mathrm{mm}^3$. Thus a total area $ld^2/t \sim 10^6\,\mathrm{mm}^2$ can be drawn. Assuming each word consists of a line ~ 5 cm long by 1/2 mm wide each word has an area of 25 mm^2, so that $\sim 50{,}000$ words (one or two hundred pages) can be written if the whole pencil is used up and you don't unnecessarily reduce the length of the pencil by over-sharpening.

2. Eyes

a) Experimentally determine the angular resolution of your eyes (in daylight, at whatever distance you have best vision, or with your usual corrective lenses in). How close are they to the diffraction limit?

The younger professor can see the spaces in 12-point text at 10 paces (25 feet), giving a resolution of $\theta_{\rm esp} \sim 0.3{\rm cm}/(25 \times 12 \times 2.5) = 4 \times 10^{-4} = 1.4$arcmin. He can also just see the 0.27/mm scan lines on his monitor from 60cm, which gives the same resolution of 4×10^{-4} radian. Measuring in a mirror in a brightly lit bathroom gives a pupil diameter of 0.5cm, so the diffraction limit in green light, $\sim \lambda/D \sim 5 \times 10^{-5}\,{\rm cm}/0.5\,{\rm cm} \sim 10^{-4}$. Thus his eyes are about a factor of 4 worse than diffraction-limited optics.

b) Use your result in (a) to estimate an upper limit to the size of the light-sensitive cells in your retina (the cones in daylight), and hence a lower limit to the total number of such cells in your eye.

The eyeball is about $L = 2$ cm diameter, so we assume that is the lens's focal length. The images formed by the lens on the retina of objects separated in angle by $\theta_{\rm esp} = 4 \times 10^{-4}$ radian will then be $d_{\rm min} \sim L\theta_{\rm esp} = 8 \times 10^{-4}$ cm $= 8\mu$m apart, so the cones must be at least this closely spaced (diffraction-limited eyes would require cones 2μm apart). Our eyes see roughly half a hemisphere, so cones must cover about $A = (1/2)4\pi(L/2)^2 = 6\,{\rm cm}^2$ of the retina, so if they were everywhere spaced by $d_{\rm min}$, there would be $> A/d_{\rm min}^2 \sim 10^7$ cones in the eye. Actually there are $\sim 6 \times 10^6$ cones, spaced by about 2μm in the central 0.3mm field, and more widely as the edge of the eye is approached. There are $\sim 10^8$ rods (used in night vision), but they seem to be connected for sensitivity, not resolution.

c) If you were an astronaut in space (no depth cues!) and saw a strange object floating toward you, how close would it have to get before your stereoscopic vision would allow you to estimate its distance?

Human eyes are separated by $s \sim 6$ cm, so an object at distance D requires the muscles controlling the eyes to make them converge by an angle $\sim s/D$. This could be done by servoing the convergence angle until corresponding parts of the image fell on corresponding cones in each eye. The convergence could only be decided this way if $s/D > \theta_{\rm esp}$, i.e. if $D < s/\theta_{\rm esp} = 1.5 \times 10^4$ cm $= 150$ m. Anybody know if the eye can really do anything like this well on unfamiliar objects, especially without nearer and farther objects in the field for 'calibration'?

3. Estimate the mass of rubber liberated from car tires each year by the cars travelling along the stretch of the 210 freeway passing through Pasadena.

If all drivers were good drivers, they would obey the DMV '2 second' rule, staying $t = 2$ seconds behind the car ahead (this allows you to react to the brake lights of the car ahead and avoid hitting it *if* you decelerate at the same rate it does. If it has better brakes than you do, bang and your insurance rates go up). Then in each lane $3600/t = 1800$ cars per hour would cross a given point, independent of vehicle speed (until the speed is low enough that the distance between cars becomes less than a car length). In practice, jerks in BMWs tailgate, and the true flux can be somewhat higher *until* the tailgating causes an accident and removes one lane from the freeway for half an hour. The 210 freeway has 4 lanes in each direction (in places there are more lanes, but it is the narrow bits that determine the maximum flux). We assume that for 3 hours in the morning, the inbound lanes carry 1.5 times the maximum recommended flux, or $3 \times 4 \times 1.5 \times 1800 = 3 \times 10^4$ cars in, and the outbound lanes carry the same number out every evening. For the other 18 hours per day, experience suggests an average of about 0.2 of the max flux in all lanes, or $18 \times 8 \times 0.2 \times 1800 = 5 \times 10^4$ cars, for a total of $\sim 1.1 \times 10^5$ cars per day.

The stretch of the 210 freeway in Pasadena is about 10 miles long, so from above, it carries $\sim 10^6$ vehicle miles per day, or $\sim 4 \times 10^8$ vehicle miles per year.

A typical tire goes bald in 5×10^4 miles. During this time, the four tires lose a volume of rubber $V \sim 4 \times 1$ cm $\times$ 10 cm $\times$ 2π 30 cm $\sim 7 \times 10^3$ cc. The density of tire rubber is approximately 1.5 g/cc, giving a mass loss of 0.2 g/mi. So the $\sim 4 \times 10^8$ vehicle miles per year we estimated correspond to a loss of $\sim 10^8$ g = 100 tons of rubber per year left on the freeway and in the air to settle as crud onto city windows!

4. Would the energy of all the calories you have consumed as food be enough to eject you from the solar system?

The daily consumption of energy is about 2500 kcal/day $\approx 10^7$ J/day. For an age of 20 yrs=7000 days, this works out to 7×10^{10} J.

To escape from the solar system, you would need to escape from the earth, then escape from the sun. To escape from the earth (for $m_{person} \sim 100$ kg), you would need an energy of

$$E = \frac{GM_{\text{earth}}}{R_{\text{earth}}} m_{\text{person}} \approx 6 \times 10^9 \, \text{J}.$$

To escape from the sun's gravitational field at the earth, one would need an energy of

$$E = \frac{1}{2} \frac{GM_{\odot}}{a} m_{\text{person}} \approx 4 \times 10^{10} \, \text{J},$$

where $a = 1.5 \times 10^{13}$ cm is the distance between earth and sun, and where the factor of 1/2 comes in because the earth (and you on it) already has half of the kinetic energy needed to escape the sun (escape velocity from a circular orbit is half of that from rest). Therefore, the total energy needed is about 5×10^{10} J, so you could barely escape.

5. [Note: to answer this question, you don't have to know any thermodynamics, though you'll appreciate it more if you do. Enthalpy has units of energy.] The enthalpy per particle $h = H/N$ of a relativistic gas containing N particles in a volume V depends on:

   $s = S/N$, the entropy per particle,
   p, the pressure,
   $\hbar$, Planck's constant,
   c, the speed of light,
   m, the rest mass of the particles in the gas,
   k, Boltzmann's constant.

   a) How many independent dimensionless quantities Π_i can be formed from these 7 variables?

The cgs units for these quantities are

h (erg or gm cm^2 s^{-2})
s (erg K^{-1})
p (erg cm^{-3})
$\hbar$ (erg s)
c (cm s^{-1})
m (gm)
k (erg K^{-1}).

We have 7 variables in 4 units (cm, s, gm, K). By the Buckingham Π theorem, $7 - 4 = 3$ independent dimensionless quantities can be formed from these variables.

b) One of these is $\Pi_1 = s/k$. Find all the others, and give an expression for H in the form $H = N v_1^{\alpha_1} v_2^{\alpha_2} \phi(\Pi_1, \Pi_2, \ldots)$, where v_1 and v_2 are two of the variables listed above. The function ϕ cannot be determined from dimensional analysis alone (and in fact also depends on the particles' dimensionless spin).

Many choices are possible, but three convenient ones, linear in each of the thermodynamic variables are $\Pi_1 = s/k$, $\Pi_2 = h/(mc^2)$ and $\Pi_3 = p\hbar^3/m^4c^5$. Buckingham tells us that $f(\Pi_1, \Pi_2, \Pi_3) = 0$, so we can solve for the root $\Pi_2 = \phi(\Pi_1, \Pi_3)$: i.e. $H = Nmc^2\phi(s/k, p\hbar^3/m^4c^5)$.

c) If the gas is nonrelativistic, the speed of light is no longer a relevant variable, so c must cancel out of the equation in (b). Give the resulting equation for H, and using the thermodynamic relation $\partial H/\partial p = V$, use that equation to prove that for any nonrelativistic gas $H = (5/2)pV$.

The only form of the general equation for H in which the variable c cancels out is $H = N(p^2\hbar^6/m^3)^{1/5}\phi(\Pi_1)$. Then $\partial H/\partial p = \frac{2H}{5p} = V$. So $H = (5/2)pV$.

d) If the gas is ultrarelativistic, c will be relevant, but the rest mass of the particles shouldn't matter. Give the resulting equation for H, and prove (as in (c)) that $H = 4pV$ for any gas of ultrarelativistic particles.

The only only form of the general equation for H in which m cancels out is

$$H = N(p\hbar^3c^3)^{1/4}\phi(\Pi_1).$$

Thus $\partial H/\partial p = \frac{H}{4p} = V$. So $H = 4pV$.