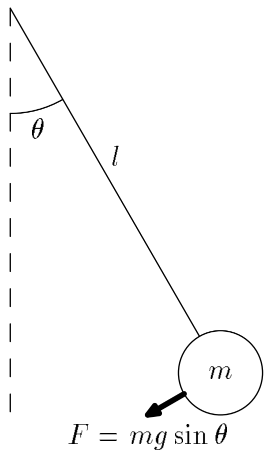
The material from the handout 'The Buckingham Pi Theorem' is not repeated here. Some examples.

Pendulum



We have a pendulum bob of mass m at the end of a rope of length l, and we want to find how the θ, the angle from the vertical, varies with time. To keep things simple, we assume the pendulum is released from rest ($\dot{\theta}_0 = 0$), at an angle θ_0. The first step is to list the variables,

$$\theta, t, m, g, l, \text{and } \theta_0, \tag{1}$$

where $\theta = f(t, m, g, L, \theta_0)$. There are 6 variables and 3 dimensions (*length*, *mass* and *time*), so $n = 6$ and $r = 3$, and there are $n - r = 3$ dimensionless Π variables. The easiest way to find them is by inspection (guess). Two of the six originals are already dimensionless, so we start with

$$\Pi_1 = \theta, \qquad \Pi_2 = \theta_0, \text{ and } \Pi_3 = ?. \tag{2}$$

Note that this choice is not unique. We might have chosen instead $\Pi_1 = \theta\theta_0$ and $\Pi_2 = \theta/\theta_0$. This Π_2 is 'more dimensionless' than θ itself, since the radians have cancelled out. But technically radians have no dimension, so both choices would work for Π_2; we will use the choices in (2), which still lacks a Π_3. This Π variable cannot depend on m since none of the other original variables in (1) has a mass unit to cancel out the mass in m. So we have g, t and l to play with, which means

$$\Pi_3 = \frac{gt^2}{l}.$$

Again, we might have chosen differently, say $\Pi_3 = \theta gt^2/l$. The Buckingham Pi theorem just tells us that here the physics can be expressed in terms of any 3 *independent* dimensionless variables. But there are an infinity of possiblities—the Pi theorem doesn't tell us which of them will give the nicest-looking equation; for that we have to use good sense. Note the emphasis on independent variables—if we didn't require that, our choices could have been $\Pi_1 = \Pi_2 = \Pi_3 = \theta$, all dimensionless.

From the zero-crossing statement of the Pi theorem, we can write the general form for θ,

$$\theta = f(\theta_0, \frac{gt^2}{l}).$$

This is as far as the Pi theorem will take us, unless we abandon our laziness and put in a bit of knowledge. From experiment, we know that f is periodic, say with period P. That is,

$$f(\theta_0, 0) - f(\theta_0, \frac{gP^2}{l}) = 0.$$

We write the allowed value of P (the root) as a function of θ_0: $\psi(\theta_0) = gP^2/l$, with ψ another dimensionless function. So we find that $P = f(\theta_0)\sqrt{l/g}$. The exact solution is

$$P = 4\sqrt{\frac{l}{g}}F\left(\frac{\pi}{2}, \sin\frac{\theta_0}{2}\right),$$

where F is an elliptic integral of the first kind. We can either look up F or determine it experimentally from small pendula with the same θ_0.

We can do some more in the limit $\theta_0 \to 0$, if we are willing to write down (but not solve) one equation. (This class truly celebrates laziness.) In that limit, the pendulum equation is

$$mgl\theta = ml^2\ddot{\theta}$$

Now we scale θ by θ_0, effectively creating the alternate choice for Π_2 θ/θ_0 which we considered above. The scaled pendulum equation is

$$mgl\frac{\theta}{\theta_0} = ml^2\frac{d^2}{dt^2}\left(\frac{\theta}{\theta_0}\right).$$

Defining $s \equiv \theta/\theta_0$,

$$mgls = ml^2\frac{d^2 s}{dt^2}, \qquad \text{with } s_0 = 1.$$

We have $n = 5$ variables,

$$m, L, g, s, \text{ and } t,$$

with $r = 3$ units, *length*, *mass*, and *time*. [Usually we have three independent units, sometimes four if we're doing electricity, sometimes less.] From the Pi theorem,

$$s = f(t, 1, l, g).$$

A short explanation of this step is in order. First, we know that none of the Π variables can contain m, for m is the only variable with *mass* in it. So f does not depend on m. That explains why t, l and g show up in the functional dependence of f. Except 1 also shows up; it is the s_0, which is by definition 1. Any function is a function of 1, so we might as well have said $f = f(t, l, g)$. From those three variables, we can form only one independent dimensionless quantity, some power of gt^2/l. [There are three variables, and three units, so we should not be able to form any Π variables. Why can we form one?] So $s = f(gt^2/l)$. As before, we argue that f must be periodic, say with period P. That is, $f(0) = f(gP^2/l)$, which gives

$$P = \Pi\sqrt{\frac{l}{g}},$$

where again Π is a dimensionless constant (which turns out to be 2π but you have to actually solve an equation to get that, which is beyond the scope of the course).

Q/ How fast do marbles fall in corn syrup, as a function of radius?

The $n = 6$ players,

R	marble radius
μ	dynamic viscosity, $\mu = \rho\nu$ where $[\mu] = M/LT$
g	acceleration due to gravity
ρ_{sphere}	marble density
ρ_{fluid}	fluid density
v	marble terminal velocity

There are $r = 3$ units, *mass*, *length*, and *time*. So we need 3 Π variables. The obvious first one is

$$\Pi_1 = \frac{\rho_{\text{sphere}}}{\rho_{\text{fluid}}}.$$

The other two turn out to be

$$\Pi_2 = \frac{v^2}{Rg} \qquad \text{and} \qquad \Pi_3 = \frac{vR\rho_{\text{fluid}}}{\mu}.$$

Aeronautics experts will recognize Π_2 as the square of the Froude number; Π_3 is the Reynolds number (abbreviated as Re). [See Tritton, *Physical Fluid Dynamics*, (Oxford: OUP, 1988) for more on these dimensionless ratios.] Again, other choice for the Π's would work.

A *question* from the audience: *how do you know you've done the job right choosing the* Π *variables? Do you have to make sure you use all the original variables?* No, that's automatically taken care of if you have an independent set of Π's. If you leave out some of the original variable, then the $n - r$ Π variables will be dependent. For example, suppose you have a problem about rectangular prisms, with variables A, B, and C as the length, width and height. There are three variables and one unit, so you need two Π variables. If you insist on not using C, you can only form one independent dimensionless variable, some power of A/B. Finding two Π's requires all the original variables.

We want to solve for v, so we use the Pi theorem in the form $\Pi_2 = f(\Pi_1, \Pi_3)$, or

$$\frac{v^2}{Rg} = f\left(\frac{vR\rho_{\text{fluid}}}{\mu}, \frac{\rho_{\text{sphere}}}{\rho_{\text{fluid}}}\right). \tag{3}$$

This is as far as we can go on total ignorance. We need a little bit of clever to march forwards. The marbles are in a thick liquid and reach terminal velocity quite quickly. Terminal velocity, by definition a constant velocity, means no net force on the marble—the gravitational and viscous forces balance. Equating the two,

$$\text{gravitational force } (\propto g) = \text{viscous drag } (\propto \mu),$$

which means $g \propto \mu$ in the final solution. Looking at the form of f in (3), we see g will be proportional to μ iff $vR\rho_{\text{fluid}}/\mu$ is a constant factor out in front of f. That is, if

$$\frac{v^2}{Rg} = \frac{vR\rho_{\text{fluid}}}{\mu} f\left(\frac{\rho_{\text{sphere}}}{\rho_{\text{fluid}}}\right).$$

Solving for v,

$$v = \frac{R^2 g\rho_{\text{fluid}}}{\mu}\psi\left(\frac{\rho_{\text{sphere}}}{\rho_{\text{fluid}}}\right),$$

where ψ is another (simpler) dimensionless function. We now need another dash of clever. If the two densities were equal ($\rho_{\text{fluid}} = \rho_{\text{sphere}}$), then the 'marble' would be neutrally buoyant and not move at all ($v = 0$). The constant g here is really a placeholder for the buoyant force; the gravitational force must appear in the solution as $g(\rho_{\text{sphere}} - \rho_{\text{fluid}})$. This can only happen if $\psi(x) = (x-1)\Pi$, where Π is a dimensionless constant. Substituting this in,

$$v = \Pi\frac{R^2 g\rho_{\text{fluid}}}{\mu}\left(\frac{\rho_{\text{sphere}}}{\rho_{\text{fluid}}} - 1\right) = \Pi\frac{R^2 g}{\mu}(\rho_{\text{sphere}} - \rho_{\text{fluid}}), \qquad \text{where } \Pi = \begin{cases} 2/9, & \text{for a sphere;} \\ 1/9, & \text{for a bubble.} \end{cases}$$

The factors for bubbles and spheres are different because they boundary between the fluid and the travelling object is different. A sphere is a separate substance, solid on the inside, while a bubble is just a gap in the fluid. The magic Π factors above couldn't be deduced by our methods of nearly total ignorance. A fluid mechanic needs to be called to do the messy work; but hopefully his or her work will be much easier since we worked everything out except one number.

In the above derivations we assumed that drag was proprtional to v, which works for $\mathrm{Re} \ll 1$. For large Re, turbulance dominates in the drag, and the viscosity drop out. So the magic f function can't depend on Π_1 because Π_1 contains the viscosity. Simplifying (3),

$$\frac{v^2}{Rg} = \psi\left(\frac{\rho_{\text{sphere}}}{\rho_{\text{fluid}}}\right) = \Pi\frac{\rho_{\text{sphere}} - \rho_{\text{fluid}}}{\rho_{\text{fluid}}}, \tag{4}$$

where Π is again some dimensionless number. Multiply both sides by $4\pi R^2/3$ (where π is the number, $3.141\ldots$), and define the coefficient of drag $c_{\mathrm{d}} \equiv 4\Pi/3$. Then we have

$$\pi\rho_{\text{fluid}} v^2 R^2 c_{\mathrm{d}} = \frac{4\pi}{3}R^3(\rho_{\text{sphere}} - \rho_{\text{fluid}})g. \tag{5}$$

In fact c_{d} depends slightly on viscosity, through the Reynold's number. We can reproduce the earlier low Re solution by assuming $c_{\mathrm{d}} \sim 1/\mathrm{Re}$ for $\mathrm{Re} \ll 1$. Empirically around $\mathrm{Re} = 1$ the drag coefficient levels out, and stays constant till 10^3, where it has a kink, and it drops sharply at 10^6 and levels off.

Q/ What is the terminal velocity for someone thrown out of an airplane?

The unfortunate victim tucks into a fetal position, making him a rough sphere, say with $R = 40\,\mathrm{cm}$. From (4), valid for high Reynold's number,

$$v^2 = \Pi Rg\frac{\rho_{\text{sphere}} - \rho_{\text{fluid}}}{\rho_{\text{fluid}}}. \tag{6}$$

A person has density close to that of water (like the earth's surface, we're 75% water), so $\rho_{\text{sphere}} \sim 1\,\mathrm{g/cm^3}$; the density of air is $\rho_{\text{fluid}} \approx 10^{-3}\,\mathrm{g/cm^3}$, and as always $g \approx\ \mathrm{cm/s^2}$. Also $\Pi = 3c_{\mathrm{d}}/4 \sim 1$ for most Reynold's numbers (from 1–10^6), so we'll drop the Π.

[A short note on how to estimate the density of air: for an ideal gas, $PV = nRT$, which if you put in $P = 1\,\mathrm{atm}$ and $T = 300\,\mathrm{K}$ you get $V/\mathrm{mole} = 22.4\,\ell$ (a number worth remembering). Air is mostly N_2, molecular weight 28, with some oxygen and carbon dioxide. So let's say 1 mole of air is $30\,\mathrm{g}$. Then the density is $30\,\mathrm{g}/22.4\,\ell \sim 1.3\,\mathrm{g}/\ell$. Another method is to remember that the atmospheric scale height is $10\,\mathrm{km}$, which is probably easier than remembering the ideal gas constant. In hydrostatic equilibrium, $p = \rho g h$, so

$$\rho = \frac{p}{gh} = \frac{10^6\ \mathrm{dyne/cm^2}}{10^3\ \mathrm{cm/s^2} \times 10^6\ \mathrm{cm}} = 10^{-3}\,\mathrm{g/cm^3}.$$

A third method is to use statistical mechanics, and assume the temperature stays constant. At height h molecules of mass m have potential energy $U = mgh$, so

$$\frac{\rho}{\rho_0} = e^{-mgh/kT},$$

where ρ_0 is the density at sea level. The scale height is where the exponent is -1, or when $h = kT/mg$. Plugging in,

$$h = \frac{kT}{mg} = \frac{kT}{mc^2}\,\frac{c^2}{g} = \frac{26\,\mathrm{meV}}{30\,\mathrm{GeV}} \times \frac{10^{21}\,\mathrm{cm}^2/\mathrm{s}^2}{1000\,\mathrm{cm/s}^2} \sim 10\,\mathrm{km},$$

where used the fact that a 'molecule' of air contains about 30 nucleons, at 1 GeV per nucleon.]

Putting the numbers into (6) and taking $\rho_{\mathrm{sphere}} - \rho_{\mathrm{fluid}} = \rho_{\mathrm{sphere}}$, we find

$$v = \sqrt{Rg\frac{\rho_{\mathrm{sphere}}}{\rho_{\mathrm{fluid}}}} = \sqrt{40\,\mathrm{cm} \times 10^3\,\mathrm{cm/s}^2 \times 1000} \sim 6\cdot 10^3\,\mathrm{cm/s} = 140\,\mathrm{mi/hr}.$$

We need to check that $\mathrm{Re} \gg 1$. The viscosity of air is $0.2\cdot 10^{-3}$ poise, so

$$\mathrm{Re} \equiv \frac{vR\rho_{\mathrm{fluid}}}{\mu} = \frac{6\cdot 10^3\,\mathrm{cm/s} \times 40\,\mathrm{cm} \times 10^{-3}\,\mathrm{g/cm}^3}{0.2\cdot 10^{-3}\,\mathrm{poise}} \sim 10^6 \gg 1.$$

The person must be careful to stay below $\mathrm{Re} = 10^6$, because of the steep drop in c_{d} above that.

Now back to the corn syrup. Experimentally, we find $v = 2\,\mathrm{cm/s}$. The relevant dimensions are

$$R \sim 0.8\,\mathrm{cm} \qquad \rho_{\mathrm{syrup}} \sim 1\,\mathrm{g/cm}^3 \qquad \rho_{\mathrm{glass}} \sim 2\,\mathrm{g/cm}^3.$$

For a sphere, the low Re result was

$$\begin{aligned} v &= \frac{2}{9}\frac{R^2 g}{\mu}(\rho_{\mathrm{sphere}} - \rho_{\mathrm{fluid}}) \\ &\sim \frac{2}{9}\frac{0.6\,\mathrm{cm}^2 \times 1000\,\mathrm{cm/s}^2}{\mu} \times 1\,\mathrm{g/cm}^3 = \frac{120}{\mu|_{\mathrm{poise}}}. \end{aligned}$$

So $\mu \sim 60\,\mathrm{poise}$.