Ph103b: Solutions to Problem Set 2

Problem 1. *Could Bill Gates buy Pasadena?* *

We will estimate the total worth of Pasadena by multiplying the total area by the price per unit area for 'improved land' (RealEstate speak for land with a zoning-approved building on it). In Pasadena, standard 1/6 acre lots with a single-family house go for \$200K–\$400K, depending on neighborhood. Mansions on 1 acre lots go for \$1M (in path of 210/710 freeway extension) to \$2.5M (near San Marino border). A reasonable average might be \$1.5M/acre. Commercial property with multi-story buildings can be worth much more (Lake Ave) or much less (E. Colorado). But for simplicity pretend the market is efficient (which it isn't because of zoning laws), so that the commercial average price/acre for improved land is the same as the residential one. The area of Pasadena is $A \sim 3\,\text{mi} \times 6\,\text{mi}$. There are 640 acres per square mile. About 20% of the land is roads, which come 'free' with your taxes. So the saleable land area is $A \sim (1-0.2) \times 18\,\text{mi}^2 \times 640\,\text{acres}/\,\text{mi} \sim 10^4$ acres, or $\boxed{\text{\$15 billion}}$. The stock market currently values Bill Gates' shares in Microsoft at about $\boxed{\text{\$20 billion}}$. So Bill might just be able to afford Pasadena, if he could liquidate his Microsoft stock without panicking the market.

Problem 2. *Estimate the mass of rubber liberated from car tires each year by the cars traveling along the stretch of the 210 freeway passing through Pasadena.*

If all drivers were good drivers, they would obey the DMV '2 second' rule, staying $t = 2$ seconds behind the car ahead (this allows you to react to the brake lights of the car ahead and avoid hitting it *if* you decelerate at the same rate it does. If it has better brakes than you do, bang and your insurance rates go up). Then in each lane $3600/t = 1800$ cars per hour would cross a given point, independent of vehicle speed (until the speed is low enough that the distance between cars becomes less than a car length). In practice, jerks in BMWs tailgate, and the true flux can be somewhat higher *until* the tailgating causes an accident and removes one lane from the freeway for half an hour. The 210 freeway has 4 lanes in each direction (in places there are more lanes, but it is the narrow bits that determine the maximum flux). We assume that for 3 hours in the morning, the inbound lanes carry 1.5 times the maximum recommended flux, or $3 \times 4 \times 1.5 \times 1800 = 3 \times 10^4$ cars in, and the outbound lanes carry the same number out every evening. For the other 18 hours per day, experience suggests an average of about 0.2 of the max flux in all lanes, or $18 \times 8 \times 0.2 \times 1800 = 5 \times 10^4$ cars, for a total of $\sim 1.1 \times 10^5$ cars per day.

The stretch of the 210 freeway in Pasadena is about 10 miles long, so from above, it carries $\sim 10^6$ vehicle miles per day, or $\sim 4 \times 10^8$ vehicle miles per year.

A typical tire goes bald in 5×10^4 miles. (Many tire companies offer 50,000-mile warranties on their tires; probably their extensive testing revealed that most of their tires died after 51,000 miles.) During this time, the four tires lose a volume of rubber $V \sim 4 \times 1\,\text{cm} \times 10\,\text{cm} \times 2\pi \times 30\,\text{cm} \sim 7 \times 10^3$ cc.

* *Historical note: In the not-too-distant past, Pasadena was owned by one person. In the years before and after 1848, when Mexico ceded California to the Union, the 14,000 acre Rancho el Rincon de San Pascual was owned by Colonel Manuel Garfias. Garfias lost title to the Rancho in 1859 to Dr. John S. Griffin and Benjamin "Don Benito" Wilson when he failed to make payments on a loan they had made him. In 1873 Wilson sold his portion to a real estate development association, which subdivided the land and named the community "Pasadena".*

The density of tire rubber is approximately 1.5 g/cc, giving a mass loss of 0.2 g/mi. So the $\sim 4\times10^8$ vehicle miles per year we estimated correspond to a loss of $\sim 10^8\,\mathrm{g} = \boxed{100\,\mathrm{tons}}$ of rubber per year left on the freeway and in the air to settle as crud onto city windows!

Problem 3. *Light bulb filaments are made of refractory metals (e.g. Tungsten) so that when heated enough to radiate at optical wavelengths they don't sublimate.*

a) *The resistance of a light bulb measured with a 3 V battery tester is about 10 times lower than it is when measured at 120 V line voltage. Why? Can you think of a consequence from your personal experience?*

b) *Predict the length and thickness of the filament of a 100 W light bulb.*

a) The resistivity is due to the scattering of electrons by lattice vibrations (Why doesn't electron-electron scattering reduce the current?). The thermal energy, kT, determines the shortest phonon wavelengths, λ:

$$kT \sim \frac{2\pi\hbar}{\lambda}, \tag{1.1}$$

since the phonon energy is $\hbar\omega = 2\pi\hbar/\lambda$. Therefore, $\lambda \propto T^{-1}$. This phonon wavelength sets the scale of the smallest lattice distortions, off of which the electrons scatter. So the electron mean free path, l_e, scales as T^{-1}. The electron velocity is $v_\mathrm{e} \sim v_\mathrm{F}$ at all (reasonable) temperatures. So the conductivity, which is proportional to $l_\mathrm{e}v_\mathrm{e}$, will scale as T^{-1}, and therefore the resistivity at room temperature ($T = 300\,\mathrm{K}$) will be ten times lower than the resistivity at light bulb temperatures, where $T \sim 3000\,\mathrm{K}$ (light bulbs are slightly yellower than the sun, which has $T \sim 6000\,\mathrm{K}$.).

The filament is at 300 K when a 3 volt battery is across it and at 3000 K under normal operation at 120 volts. Thus the power dissipated in the filament $P = V^2/R$ is 10 times higher when it is first turned on from a cold state. That is why bulbs almost always blow (with a blue flash: briefly reaches 6000 K and sublimates like mad!) when you turn them on, not while they are on steadily.

b) Symbols: P power, V voltage, R resistance, ρ resistivity, ℓ length, r cross sectional radius, T temperature, $\sigma \approx 6\times10^{-12}\,\mathrm{watts\,deg^{-4}cm^{-2}}$ (the Stefan-Boltzmann constant).

$$P = \frac{V^2}{R} = 2\pi r\ell\sigma T^4, \tag{1.2}$$

with $R = \rho\ell/\pi r^2$ yields

$$\frac{\ell}{r^2} = \frac{\pi V^2}{\rho P} \quad \text{and} \quad r\ell = \frac{P}{2\pi\sigma T^4}. \tag{1.3}$$

Solving for r and ℓ using the numerical values $\rho \sim 2\times10^{-5}$ ohm cm for the *high temperature* resistivity of tungsten, and $T \approx 3000\,\mathrm{K}$ for the operating temperature of the filament, we obtain

$$\boxed{r \sim 2\times10^{-3}\,\mathrm{cm}}, \quad \text{and} \quad \boxed{\ell \sim 20\,\mathrm{cm}}. \tag{1.4}$$

Sterl has verified this by smashing a General Electric 100 W, 120 V lightbulb. To the naked eye, the coiled filament looked much thicker than the estimate above. However a microscope shows that what the naked eye interprets as the filament is in fact a very thin coil of wire, the wire being about 40 μ in diameter. Thus the filament is a coiled coil, with 20 little coils (diameter 200 μ) per big coil (diameter 0.1 cm), and 34 big coils in the filament, for a total length of $34 \times 20 \times \pi \times 2 \times 10^{-2}$ cm = 40 cm. Order of magnitude physics triumphs again!

Problem 4. *In class, by considering the energy of extra bonds per neighbor at the surface, we estimated the surface tension of a liquid.*

a) *Using the same procedure and what you know of nuclear binding energies, estimate the surface tension of the nuclear fluid. The radius of an atomic nucleus of atomic number A is $\sim 1.2 \times 10^{-13} A^{1/3}$ cm.*

b) *For heavy nuclei, considerations of nuclear physics require that the number of protons Z is given by $Z/A \sim 0.4$. Show that the electrostatic plus surface energy of nuclei with $A < A_{\max}$ is increased by fission (splitting the nucleus into two halves), so fission will not occur spontaneously, but that for $A > A_{\max}$ the energy is reduced by spontaneous fission, and estimate $A_{\max}$. Your answer will be improved if you estimate the electrostatic energy carefully.*

c) *Estimate the maximum possible angular momentum of a tin nucleus (mass number $A = 120$, atomic number $Z = 50$), in units of $\hbar$.*

a) We will do this problem more carefully than usual, since we are trying to estimate a dimensionless number to order of magnitude! The total binding energy of a nucleus in the liquid drop picture is made up of a bulk energy per nucleon U_B minus a surface tension energy $U_s = 4\pi r^2 \gamma$, and minus the electrostatic (repulsion) energy of the protons, $U_e = (3/5)Q^2/r$ for a uniformly charged sphere of radius r and charge Q. Here $Q = Ze = 0.4Ae$ and $r = 1.2A^{1/3}$ fm. This gives $U_e = 0.12A^{5/3}$ MeV. In class, we showed that the surface tension of a liquid $\gamma \simeq U_B/(6a^2)$, where a was the interparticle distance. For the nucleus $A = (4\pi/3)(r/a)^3$, so using the expression for r, we find $a = 1.9$ fm and $U_s = 0.83 U_B A^{2/3}$. Thus we have the total binding energy of a nucleus of atomic number A:

$$E(A) = U_B A - 0.83 U_B A^{2/3} - 0.12\,\mathrm{MeV}\, A^{5/3}\,. \tag{1.5}$$

For iron ($A = 56$), the most tightly bound nucleus, it is well known that $E/A = 8.8$ MeV, so1.5 gives $U_B = 14$ MeV and a surface energy

$$U_s = 4\pi r^2 \gamma \simeq \boxed{10 A^{2/3}\,\mathrm{MeV}}\,. \tag{1.6}$$

The actual best-fit mass formula has $U_B = 16$ MeV and $U_s = 18A^{2/3}$ MeV, and the electrostatic term exactly as we have it.

b) Fission will occur if the two halves are more bound than the whole:

$$2E(A/2) > E(A). \tag{1.7}$$

Note that the bulk energy is the same for the two fission halves as the original nucleus, so only the surface tension term (which prefers large nuclei) and the electrostatic term (which prefers small, less charged nuclei) matter. Thus using1.5 the condition is

$$2[0.83 U_B (A/2)^{2/3} + 0.12\,\mathrm{MeV}(A/2)^{5/3}] < 0.83 U_B A^{2/3} + 0.12\,\mathrm{MeV}\, A^{5/3} \tag{1.8}$$

or

$$0.22 U_B A^{2/3} < 0.044 A^{5/3}\,\mathrm{MeV} \tag{1.9}$$

or $\boxed{A > 70}$. Actually spontaneous fission (at detectable rates) starts with the uranium isotopes $A = 234 - 238$. But note that even for these, fission decay is very rare: more than 99.99% of the decays occur by emission of an α particle (^{4}He nucleus), which is more tightly bound and has a much larger phase space for decay.

c) A nucleus is held together by the strong interactions between nucleons, and destabilized by the Coulomb repulsion of the protons and the centrifugal force due to rotation. Since tin is far from the fission limit in the absence of rotation, it should be possible to neglect the Coulomb force. Thus, the question becomes: How much rotational energy does the nucleus need in order to create the extra surface that appears when it is split into two nuclei of one half its volume? [This is similar to the argument used in class to compute the maximum size of a rain drop falling at terminal velocity, with centrifugal force splitting the drop instead of air drag.] The mass of the nucleus is $M = Am_p$ and its radius is $R = 1.2 \times 10^{-13} A^{1/3}$ cm, thus the rotational energy is $(\hbar l)^2/2I = 36 l^2 A^{-5/3}$ MeV, where $I = (2/5)MR^2$ is the moment of inertia, and l is the angular momentum in units of $\hbar$. Using our estimate from part (a), the surface energy of a nucleus is $\sim 10A^{2/3}$ MeV (as mentioned in part (a), the best fit mass formula gives 18 instead of 10), so the energy cost for splitting it in two is $10(2 \times (A/2)^{2/3} - A^{2/3})$ MeV $\sim 3A^{2/3}$ MeV. Thus, the maximum angular momentum is

$$l_{crit} \sim (3/36)^{1/2} A^{7/6} = 0.27 A^{7/6} \approx \boxed{60} \tag{1.10}$$

for $A = 120$ (we would have gotten $l_{crit} = 95$ if we'd used the best-fit surface tension energy $18A^{2/3}$ MeV of the mass formula).

A. J. Sierk (Phys. Rev. **C 33**, 2039 (1986)) worked out a more sophisticated model for rotating nuclei that is claimed to agree well with the experimental data. One of his graphs gives the values $l_{\mathrm{I}} \approx 68$ (the angular momentum above which there is no axisymmetric equilibrium state) and $l_{\mathrm{II}} \approx 83$ (the angular momentum above which there is *no* equilibrium state). So order of magnitude did pretty well here.

Problem 5. *[Note: to answer this question, you don't have to know any thermodynamics, though you'll appreciate it more if you do. Enthalpy has units of energy.] The enthalpy per particle $h = H/N$ of a relativistic gas containing N particles in a volume V depends on:*

$s = S/N$, the entropy per particle,
p, the pressure,
$\hbar$, Planck's constant,
c, the speed of light,
m, the rest mass of the particles in the gas,
k, Boltzmann's constant.

[Why not temperature T, density $n = N/V$, etc. too? Because any two variables suffice to specify the thermodynamic state of a system. For use with h, s and p are the two natural variables, but (almost) any other pair would do]

a) How many independent dimensionless quantities Π_i can be formed from these 7 variables?

b) One of these is $\Pi_1 = s/k$. Find all the others, and give an expression for H in the form $H = N v_1^{\alpha_1} v_2^{\alpha_2} \phi(\Pi_1, \Pi_2, \ldots)$, where v_1 and v_2 are two of the variables listed above. The function ϕ cannot be determined from dimensional analysis alone (and in fact also depends on the particles' dimensionless spin).

c) If the gas is nonrelativistic, the speed of light is no longer a relevant variable, so c must cancel out of the equation in (b). Give the resulting equation for H, and using the thermodynamic relation $\partial H/\partial p = V$, use that equation to prove that for any nonrelativistic gas $H = (5/2)pV$.

d) If the gas is ultrarelativistic, c will be relevant, but the rest mass of the particles shouldn't matter. Give the resulting equation for H, and prove (as in (c)) that $H = 4pV$ for any gas of ultrarelativistic particles.

a) The cgs units for these quantities are

h	(erg or gm cm^2 s^{-2})
s	(erg K^{-1})
p	(erg cm^{-3})
$\hbar$	(erg s)
c	(cm s^{-1})
m	(gm)
k	(erg K^{-1})

We have 7 variables in 4 units (cm, s, gm, K). By the Buckingham Π theorem, $7 - 4 = \boxed{3}$ independent dimensionless quantities can be formed from these variables.

b) Many choices are possible, but three convenient ones, linear in each of the thermodynamic variables are $\boxed{\Pi_1 = s/k}$, $\boxed{\Pi_2 = h/(mc^2)}$ and $\boxed{\Pi_3 = p\hbar^3/m^4c^5}$. Buckingham tells us that

$$f(\Pi_1, \Pi_2, \Pi_3) = 0, \tag{1.11}$$

so we can solve for the root $\Pi_2 = \phi(\Pi_1, \Pi_3)$: i.e. $\boxed{H = Nmc^2\phi(s/k, p\hbar^3/m^4c^5)}$.

c) The only form of the general equation for H in which the variable c cancels out is

$$\boxed{H = N(p^2\hbar^6/m^3)^{1/5}\phi(\Pi_1)}. \tag{1.12}$$

Then $\partial H/\partial p = 2H/5p = V$. So $\boxed{H = (5/2)pV}$.

d) The only only form of the general equation for H in which m cancels out is

$$\boxed{H = N(p\hbar^3c^3)^{1/4}\phi(\Pi_1)}. \tag{1.13}$$

Thus $\partial H/\partial p = \frac{H}{4p} = V$. So $\boxed{H = 4pV}$.