Ph103b: Solutions to Problem Set 9

Problem 1. *Writing with Pencils*

a) *Perform an experiment to estimate the thickness, in atoms, of the graphite layer left by a pencil writing on a piece of paper.*

b) *Provide a theoretical estimate based on the material presented in class.*

a) 4000 mm of 0.5 mm pencil line required 1 mm of lead (using 0.5 mm × 0.5 mm lead). So the thickness of the layer is

$$t \sim \frac{\textit{Volume}}{\textit{Area}} \sim \frac{0.5\,\mathrm{mm} \times 0.5\,\mathrm{mm} \times 1\,\mathrm{mm}}{4000\,\mathrm{mm} \times 0.5\,\mathrm{mm}} \sim 1000\,\text{Å}, \tag{1.1}$$

or about $\boxed{300 \text{ layers}}$.

b) Graphite is black: it must absorb in most of the visible spectrum. The integrated harmonic oscillator cross-section, $\int \sigma(\nu)\, d\nu$, is $\pi f e^2/mc$, where f is the oscillator strength. Using the usual boxcar absorption shape, we can approximate the cross-section in the absorption band:

$$\sigma \sim \pi f \frac{e^2}{mc} \frac{1}{\Delta\nu}, \tag{1.2}$$

where $\Delta\nu \sim \nu_{\text{violet}} - \nu_{\text{red}} \sim \nu_{\text{red}}$ is the width of the boxcar. Using $e^2/mc^2 = r_0 = \alpha^2 a_0$, where r_0 is the classical electron radius, we have

$$\sigma \sim \pi f \alpha^2 a_0 \lambda_{\text{red}}. \tag{1.3}$$

Putting in numbers,

$$\sigma \sim 3 \times f \times \frac{1}{2} \cdot 10^{-4} \times 0.5 \cdot 10^{-8}\,\mathrm{cm} \times 7 \cdot 10^{-5}\,\mathrm{cm} \sim 5 \cdot 10^{-17} f\,\mathrm{cm}^2. \tag{1.4}$$

Graphite layers are lattices of benzene rings without the hydrogens. Say each benzene ring contributes one scattering electron. Each carbon belongs to three rings; each ring has six carbons, so if each ring has one scattering electron, then each carbon provides half a scattering electron. The density of graphite is roughly $2.5\,\mathrm{g\,cm^{-3}}$, so the number density of scattering electrons is

$$n \sim 0.5 \times \frac{2.5\,\mathrm{g}}{\mathrm{cm}^3} \times \frac{6 \cdot 10^{23}}{12\,\mathrm{g}} \sim 6 \cdot 10^{22}\,\mathrm{cm}^{-3}. \tag{1.5}$$

Ring electrons, as in graphite, are difficult to polarize if the incident light is parallel to the plane of the ring; scattering occurs only for light incident reasonably perpendicular to the plane of the ring, say within ~ 1 radian. So we'll take $f \sim 1/2\pi \sim 0.1$. The mean free path is then

$$l \sim (n\sigma)^{-1} \sim \left(6 \cdot 10^{22}\,\mathrm{cm}^{-3} \times 0.1 \times 5 \cdot 10^{-17}\,\mathrm{cm}^2\right) \sim 3 \cdot 10^{-6}\,\mathrm{cm} = 300\,\text{Å}. \tag{1.6}$$

This distance is about 100 layers. The light goes through the graphite twice (incidence and reflection), so 100 layers produce an optical depth of 2; maybe we want an optical depth of 4 to make a black mark, so we'll estimate $\boxed{200 \text{ layers}}$ (with good agreement with part a).

Problem 2. *Consider a mass $m = 1kg$ hung from a massless string of length $\ell = 1m$ (a simple pendulum). Estimate the rms angular displacement of the string from vertical due to:*

a) quantum fluctuations,

b) thermal fluctuations at $T = 300K$.

a) The mass sits in a harmonic oscillator potential and has ground state energy $E_0 \sim \hbar\omega_0$, where $\omega_0 = \sqrt{g/l}$. An angular displacement θ requires potential energy $\sim mg\theta^2 l = m\omega_0^2 l^2\theta^2$, so the energy E_0 corresponds to

$$\theta^2 \sim \frac{E_0}{m\omega_0^2 l^2} \sim \frac{\hbar}{m\omega_0 l^2}. \tag{1.7}$$

So

$$\theta_{\rm rms} \sim \left(\frac{\hbar}{m\omega_0 l^2}\right)^{1/2}. \tag{1.8}$$

The resonant frequency is $\omega_0 = \sqrt{10\,{\rm m\,s^{-2}}/1\,{\rm m}} \sim 3\,{\rm s}^{-1}$, so

$$\theta_{\rm rms} \sim \left(\frac{10^{-34}\,{\rm J\,s}}{1\,{\rm kg} \times 3\,{\rm s}^{-1} \times 1\,{\rm m}^2}\right)^{1/2} \sim \boxed{5 \cdot 10^{-18}\,{\rm rad.}} \tag{1.9}$$

b) Instead of the zero-point energy E_0 in1.7, we use $E \sim kT$ for the energy. Then,

$$\theta_{\rm rms} \sim \left(\frac{kT}{mw_0^2 l^2}\right)^{1/2} \sim \left(\frac{0.025\,{\rm eV} \times 1.6 \cdot 10^{-19}\,{\rm J/eV}}{1\,{\rm kg} \times 10\,{\rm s}^{-2} \times 1\,{\rm m}^2}\right)^{1/2} \sim \boxed{2 \cdot 10^{-11}\,{\rm rad.}} \tag{1.10}$$

Problem 3. *Free-Free Absorption*

a) Derive an approximate analytic formula giving the absorption length for an electromagnetic wave of radian frequency ω propagating in a completely ionized hydrogen plasma of number density n and temperature T. Proceed by calculating the energy stored in the oscillations of the electrons forced by the incident radiation. Assume that this energy is dissipated as heat during strong electron-proton collisions during which the electron is deflected by an angle of order a radian or greater.

b) A massive star is surrounded by a sphere of ionized hydrogen of of radius $R = 10^{18}$ cm with $n = 10^3\,{\rm cm}^{-3}$, $T = 10^4$ K. What is the critical wavelength λ above which the sphere is opaque to electromagnetic waves?

a) An electric field E oscillating at frequency ω accelerates the electrons with acceleration $a \sim eE/m_e$ for a time $\tau \sim 1/\omega$. So the electrons acquire a velocity $v \sim a\tau \sim eE/m_e\omega$. The kinetic energy per electron is $\sim m_e v^2$ and the kinetic energy density is $\mathcal{KE} \sim m_e n v^2$, or

$$\mathcal{KE} \sim \frac{ne^2}{m_e\omega^2}E^2 \sim \left(\frac{\omega_{\rm p}}{\omega}\right)^2 E^2 \sim \left(\frac{\omega_{\rm p}}{\omega}\right)^2 \mathcal{E}_{\rm wave}, \tag{1.11}$$

where $\omega_{\rm p}$ is the plasma frequency, $(4\pi ne^2/m_e)^{1/2}$, and $\mathcal{E}_{\rm wave} = E^2/4\pi$ is the energy density in the electromagnetic wave. [Equation1.11 illustrates the meaning of the plasma frequency. Below the plasma frequency, $\mathcal{KE} > \mathcal{E}_{\rm wave}$. So the wave would have to put more kinetic energy into the electrons than it actually has; therefore the wave can't propagate.]

We have to estimate how quickly this kinetic energy is dissipated in the strong electron–proton collisions. In a strong collision, where the electron is deflected by ~ 1 radian, the electron no longer has its extra energy from the wave aligned along the electric field direction. So the extra energy from the wave goes away in a single strong collision.

We also need to know the rate at which these strong collisions happen. To deflect the electron significantly, the electrostatic force must provide an impulse $\Delta p \sim m_e v_e$. If the distance of closest approach is b (roughly the impact parameter), then the force acts strongly while the electron is within b of the proton, or for a time $t \sim b/v_e$. Thus, we require $\Delta p \sim e^2/b^2 t \sim e^2/bv_e$ to be about $m_e v_e$, or $b \sim e^2/m_e v_e^2$. Equivalently, we could just say that the electrostatic potential energy, e^2/b, is approximately the kinetic energy, $m_e v_e^2$, so once again, $b \sim e^2/m_e v_e^2$. Therefore the strong-collision cross-section is

$$\sigma \sim b^2 \sim \left(\frac{e^2}{m_e v_e^2}\right)^2 \sim \left(\frac{e^2}{kT}\right)^2. \tag{1.12}$$

The collision rate is $R \sim n\sigma v_e$, so the damping rate for the kinetic energy given to the plasma is $\Gamma_{\mathrm{KE}} \sim R$. The damping rate for the wave energy is a factor of $(\omega/\omega_{\mathrm{p}})^2$ less than Γ_{KE} because the wave only donates $(\omega/\omega_{\mathrm{p}})^2$ of its energy to kinetic energy in the electrons. So the damping rate is

$$\Gamma \sim \left(\frac{\omega_{\mathrm{p}}}{\omega}\right)^2 \Gamma_{\mathrm{KE}} \sim \left(\frac{\omega_{\mathrm{p}}}{\omega}\right)^2 \left(\frac{e^2}{kT}\right)^2 \left(\frac{kT}{m_e}\right)^{1/2} n. \tag{1.13}$$

The absorption length is

$$\boxed{l \sim c/\Gamma \sim \left(\frac{\omega}{\omega_{\mathrm{p}}}\right)^2 \left(\frac{kT}{e^2}\right)^2 \left(\frac{m_e c^2}{kT}\right)^{1/2} n^{-1}.} \tag{1.14}$$

We put in numbers ($kT \sim 1\,\mathrm{eV}$ and $n = 10^3\,\mathrm{cm}^{-3}$) and use $e^2 = \hbar c\alpha = 1.5 \cdot 10^{-7}\,\mathrm{eV\,cm}$, to find that

$$l \sim \left(\frac{\omega}{\omega_{\mathrm{p}}}\right)^2 \left(\frac{1\,\mathrm{eV}}{1.5 \cdot 10^{-7}\,\mathrm{eV\,cm}}\right)^2 \times 0.7 \cdot 10^3 \times 10^{-3}\,\mathrm{cm}^3 \sim \left(\frac{\omega}{\omega_{\mathrm{p}}}\right)^2 3 \cdot 10^{13}\,\mathrm{cm}. \tag{1.15}$$

We want $l \sim 10^{18}\,\mathrm{cm}$ (or more), so we take $\omega > 200\omega_{\mathrm{p}}$. The plasma frequency is (using $e^2/m_e = a_0(\alpha c)^2$):

$$\omega_{\mathrm{p}} \equiv \left(\frac{4\pi n e^2}{m_e}\right)^{1/2} \sim \left(12 \times 10^3\,\mathrm{cm}^{-3} \times 0.5 \cdot 10^{-8}\,\mathrm{cm}\right)^{1/2} \times 0.01 \times 3 \cdot 10^{10}\,\mathrm{cm\,s}^{-1} \sim 2 \cdot 10^6\,\mathrm{s}^{-1}. \tag{1.16}$$

Thus, $\omega > 4 \cdot 10^8\,\mathrm{s}^{-1}$ or $f > 7 \cdot 10^7\,\mathrm{Hz}$. The critical wavelength is $\lambda \sim c/f \sim \boxed{4\,\mathrm{m}}$.

Problem 4. *Piano strings*

a) *The highest note on a grand piano (C8, 4186Hz) is a steel string of 5 cm length. Estimate the strain ϵ of this string. Are you impressed by the quality of steel required? Increasing the string tension increases the forces exerted on the sound board, and thus the maximum loudness of the piano.*
b) *What is the Mach number (in air) of the transverse waves on the piano wire of (a)?*
c) *The string of the lowest note (A1, 27.5Hz) on the grand piano is 2m in length; it consists of a steel core under tension, surrounded by a copper winding (under no tension) with diameter double that of the steel core. Estimate the strain ϵ of this string. Only the strings of the lowest 20 notes on the piano are wound in this way. Can you see why?*

d) *The upper strings are coupled to the lower ones through the pins and soundboard, so when a low string is struck, all upper strings resonant with its harmonics vibrate sympathetically. To minimize beating between these sympathetic vibrations of the upper strings (or their primary vibrations if they are played as part of a chord!) and the overtones of the lower strings (recall the anharmonicity due to stiffness), the notes below middle C are tuned progressively flatter, and notes above tuned progressively sharper (Railsback curve). In a properly tuned piano, C8 is tuned a factor 1.017 higher than the 16th harmonic of C4 (middle C). Estimate the diameter to length ratio a/L and the string tension (in kg or pounds force) of the middle C string, which is about 80cm long.*

a) On a string of length l, the fundamental has wavelength $\lambda = 2l$, and frequency $\nu = c_T/\lambda$. The speed of sound in steel is

$$c_{\rm s} \sim \sqrt{\frac{\mathcal{M}}{\rho}} \sim \left(\frac{2\cdot 10^{12}\,\mathrm{dyne\,cm^{-2}}}{8\,\mathrm{g\,cm^{-3}}}\right)^{1/2} = 5\cdot 10^5\,\mathrm{cm\,s^{-1}} = 5\,\mathrm{km\,s^{-1}}. \tag{1.17}$$

The transverse wave speed is, as we found in problem set 8 (problem 3a), $c_T = \epsilon^{1/2} c_{\rm s}$. So $\nu = \epsilon^{1/2} c_{\rm s}/2l$. Solving for ϵ and putting in numbers, we find that

$$\epsilon = \left(\frac{2l\nu}{c_{\rm s}}\right)^2 \sim \left(\frac{2\times 5\,\mathrm{cm}\times 4186\,\mathrm{Hz}}{5\cdot 10^5\,\mathrm{cm\,s^{-1}}}\right)^2 \sim 0.007. \tag{1.18}$$

The steel is high quality: 0.006 is the breaking strain of the strongest of the 40 steels listed in the AIP handbook (it is double the breaking strain of typical steel, and several times the yield strain).

b) The transverse wave speed is $4186\,\mathrm{Hz}\times 0.1\,\mathrm{m} \sim \boxed{420\,\mathrm{m\,s^{-1}}}$, which is about $\boxed{\text{Mach } 1.3}$ in air. As noted in solution set 8 (problem 3a), the transverse wave speed is also given by $c_T = \sqrt{\epsilon} c_{\rm s}$, where $c_{\rm s} \sim \sqrt{\mathcal{M}/\rho}$ is the speed of compressional waves in steel. Since $c_{\rm s} \sim 5\,\mathrm{km\,s^{-1}} \sim 15 c_{\rm s}^{\rm air}$, with strains of order $15^{-2} \sim 0.005$, the transverse waves will be barely supersonic (in air).

c) The winding increases the mass per length, μ, by roughly a factor of four, but doesn't change the tension, so $c_T = \sqrt{T/\mu}$ falls by a factor of two. Therefore $c_T = \epsilon^{1/2} c_{\rm s}/2$, and instead of1.18, we have

$$\epsilon = 4\left(\frac{2l\nu}{c_{\rm s}}\right)^2 \sim 4\times\left(\frac{2\times 200\,\mathrm{cm}\times 27.5\,\mathrm{Hz}}{5\cdot 10^5\,\mathrm{cm\,s^{-1}}}\right)^2 \sim \boxed{0.002.} \tag{1.19}$$

Above $\sim 100\,\mathrm{Hz}$ (about 24 notes above A1), an unwrapped string will fit in less than roughly 2 m with no problem (even at the maximum strain $\epsilon \sim 0.007$). So there's no need to wrap them to reduce c_T.

d) From class, the frequency, including stiffness, is given by

$$\omega_n^2 = \left(\frac{n\pi}{l}\right)^2 \left(\frac{T}{\rho ab}\right) + \frac{\mathcal{M}a^2}{12\rho}\left(\frac{n\pi}{l}\right)^4. \tag{1.20}$$

We will assume that the second term (the stiffness correction) is small relative to the first term. Then

$$\omega_n \approx \frac{n\pi}{l}\sqrt{\frac{T}{\rho ab}}\left\{1 + \frac{1}{24}\frac{\mathcal{M}a^3 b}{T}\left(\frac{n\pi}{l}\right)^2\right\}. \tag{1.21}$$

The factor in front is frequency of the unperturbed nth harmonic, since $\sqrt{T/\rho ab} = c_T$. So, using $T = \epsilon \mathcal{M} ab$, we find

$$f_n = f_n^0 \left\{ 1 + \frac{1}{24} \left(n \frac{a}{l} \frac{\pi}{\sqrt{\epsilon}} \right)^2 \right\}. \tag{1.22}$$

The actual frequency of the 16th harmonic of C4 is a factor 1.017 higher than it would be without stiffness (C8 is tuned sharp, so that it matches the actual 16th harmonic of C4). So from1.22, we find that

$$0.017 \sim \frac{1}{24} \left(n \frac{a}{l} \frac{\pi}{\sqrt{\epsilon}} \right)^2, \tag{1.23}$$

where $n = 16$. To solve for a/l, we need to find ϵ. The length of the string is $l = 80\,\mathrm{cm}$ and its frequency is $\nu \sim 262\,\mathrm{Hz}$. From1.18, we have $\epsilon \sim (2 \times 80 \times 262/5 \cdot 10^5)^2 \sim 0.007$. Substituting $\epsilon = 0.007$ and $n = 16$ into1.23, we find $\boxed{a/l \sim 0.001}$. Therefore the string has width $a \sim 0.8\,\mathrm{mm}$, which seems about right.

The tension is

$$T \sim \epsilon \mathcal{M} a^2 \sim 0.007 \times 2 \cdot 10^{12}\,\mathrm{dyne\,cm^{-2}} \times 0.6 \cdot 10^{-2}\,\mathrm{cm}^2 \sim 8 \cdot 10^7\,\mathrm{dyne} \sim \boxed{80\,\text{kg-force.}} \tag{1.24}$$

Fletcher & Rossing, *The Physics of Musical Instruments*, (New York: Springer-Verlag, 1991) say that all the piano strings are at about 180 lb-force, or 82 kg-force, each. (Sometimes you get lucky with order-of-magnitude.)

Problem 5. *Generation of Sound by Turbulence*

a) *Consider three-dimensional fluid turbulence with characteristic velocity v, outer scale L (the scale on which the turbulent motions are driven), and Mach number $M \sim v/c_s \ll 1$. What is the approximate amplitude of the turbulent pressure fluctuations?*

b) *Estimate the efficiency of acoustic radiation by the turbulence. Express the power radiated per unit volume as a fraction of the total energy dissipation rate per unit volume, $\mathcal{E} \sim \rho v^3/L$.*
Hint: Quadrupoles are the lowest order acoustic multipoles for free turbulence (can you see why?).

a) From Bernoulli, $\rho v^2 + p$ is a constant along a streamline. So velocity fluctuations $\sim v$ are caused by pressure fluctuations $\Delta p \sim \rho v^2$. In general, $c_\mathrm{s}^2 = p/\rho$, so

$$\boxed{\Delta p \sim p(v/c_\mathrm{s})^2 = pM^2.} \tag{1.25}$$

For gases, the p is the gas pressure. For liquids, it is the bulk modulus.

b) Sound is radiated because of the motion of the turbulent eddies. The frequency is set by how often the eddies cross the length scale, l, so $\omega \sim v/l$. From the lecture on sound (or the Pi theorem), the power radiated by a monopole of size l is

$$P_\mathrm{mono} \sim \rho \frac{\omega^2 v^2 l^4}{c_\mathrm{s}}. \tag{1.26}$$

Since $\omega \sim v/l$,

$$P_\mathrm{mono} \sim \rho \frac{v^4 l^2}{c_\mathrm{s}}. \tag{1.27}$$

In terms of the Mach number,

$$P_{\text{mono}} \sim \rho M^4 \frac{l^2}{c_s^3}, \tag{1.28}$$

where $M \equiv v/c_s$.

Mass injection (*e.g.*, a pulsing sphere) generates monopole radiation; momentum injection (*e.g.*, a car moving through air) generates dipole radiation. Free turbulence has neither mass nor momentum injection (that's what the 'free' means), so the lowest allowed multipole is quadrupole.

We'll assume that quadrupole radiation is the first allowed multipole. For a dipole, the pressure fluctuations are multiplied by the separation, l over the size of the near zone, c_s/ω:

$$\frac{\Delta p_{\text{dipole}}}{\Delta p_{\text{monopole}}} \sim \frac{l}{c_s/\omega} \sim \frac{v}{c_s} \equiv M. \tag{1.29}$$

For a quadrupole, the factor is M^2. Since power is quadratic in the pressure fluctuations, the power in1.28 needs to be scaled by M^4:

$$P_{\text{quad}} \sim \rho M^8 l^2 c_s^3. \tag{1.30}$$

The power density dissipated in the turbulence is $\epsilon \sim \rho v^3/l = \rho M^3 c_s^3/l$, and in a volume l^3 it is

$$P_{\text{turbulence}} \sim \rho v^3 l^2 = \rho M^3 l^2 c_s^3. \tag{1.31}$$

The efficiency of acoustic radiation is the ratio of1.30 to1.31:

$$\textit{Efficiency} \sim \frac{\rho M^8 l^2 c_s^3}{\rho M^3 l^2 c_s^3} = \boxed{M^5.} \tag{1.32}$$

Moral: fast turbulent flows are noisy. In most flows, where for example dipole radiation is important, the power law isn't so steep (M^3 instead), but speed still counts.