Solutions to Problem Set 3

1. Electromagnetic communication with submarines is accomplished via low frequency (5-100 hertz) radio waves. What is the penetration depth for the radio waves below the surface of the ocean at 5Hz and at 100Hz? The resistivity of seawater is on Purcell's sheet.

   From Maxwell's equations in Gaussian units, we find that **E** and **B** obey the equation

$$\left\{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{4\pi\sigma}{c^2}\frac{\partial}{\partial t}\right\}\left\{\begin{matrix}\mathbf{E}\\ \mathbf{B}\end{matrix}\right\} = 0.$$

   which implies the dispersion relation

$$k^2 = \frac{\omega^2}{c^2} + \frac{4\pi\sigma\omega}{c^2}i$$

   for plane waves with the spatial behavior e^{ikz}. In Gaussian units, the conductivity of seawater is $\sigma \sim 9 \times 10^{11}/25\ \mathrm{s}^{-1} \sim 3.6 \times 10^{10}\ \mathrm{s}^{-1}$ which is much larger than $\omega = 2\pi\nu$. Thus the second term on the right hand side of the dispersion relation is much larger than the first, and the penetration depth of the field amplitude is given by

$$\delta \equiv \frac{1}{\mathrm{IM}\ (k)} \approx \frac{c}{2\pi}\frac{1}{(\nu\sigma)^{1/2}},$$

   or about 100 m at 5 Hz and only 20 m at 100 Hz. US nuclear subs operate at 500m depth. Soviet Alfa class subs (titanium alloy hull) can reach 840m depth. So subs may have to come closer to the surface to communicate. Notice that the wavlengths used are of order the earth's radius. Thus at particular frequencies the earth and the ionosphere form a resonant cavity. At those frequencies, relatively low-power transmitters (with sizes of order the state of Wisconsin) can build up high intensity standing waves in the cavity. Every little bit helps. Note: an attempt to solve this problem purely by dimensionless analysis would be frustrated because σ/ω is dimensionless. However, if one adds the fact that static (nearly static in this case) electromagnetic fields suffer ohmic diffusion, then it is clear that δ must be proportional to $\sigma^{-1/2}$.

2. The water in a Japanese bath is about $\Delta T \approx 6$ degrees centigrade hotter than body temperature. When submerged up to your neck in such a bath,

   a) at what rate (in watts) does heat flow into your body provided that:

      i) you move around at 1 meter per second,

      ii) you remain motionless for 5 minutes?

   b) how does your body manage to maintain its temperature at a safe level?

   Useful information: The Prandtl number for water, $Pr \equiv \nu/\kappa \approx 6$, where ν is the kinematic viscosity and κ is the thermal diffusivity.

a)

i) A viscous boundary layer of thickness $\delta_\nu \sim (\nu L/v)^{1/2} \sim 1$mm develops (in the water). The thermal boundary layer has thickness, $\delta_\kappa \sim (\kappa/\nu)^{1/3}\delta_\nu \sim 0.6$mm. The energy flowing into your body, through surface area A, would be $dE/dT \sim \rho c_p \kappa \Delta T A/\delta_\kappa \sim 6$kW, a prodigious value.

ii) In this case, a thermal boundary layer of thickness $\delta_{\rm th} \sim \sqrt{\kappa t} \sim 7$mm limits the flux to about 600 W, an energy release typical of fairly vigorous exercise.

b) Evaporative cooling is the only efficient method for removing such a quantity of heat. It would have to be done through the only part of the body sticking out of the water - the head. Of course, the above heat loads are upper limits since the temperature rise of the body, particularly the skin, has been neglected.

3. Rare earth alloy magnets have permanent magnetisations corresponding roughly to the permanent alignment of a few electron spins (magnetic moment=Bohr magneton) per atom. If two such magnets, roughly cubical in shape, are allowed to pull each other together from a large separation, what is the ratio of their kinetic energy at the moment of impact to their total atomic binding energy [hint: you should be able to express your answer just in terms of powers of the fine structure constant, and dimensionless factors of order unity]? Do you think they are likely to break when they hit?

   Force between pairs w/ dipole moment m is $2m^2/r^4$, so energy when hit is $2m^2/L^3$, where L is characteristic size of magnet. $m = n\mu L^3$, where n is number density of aligned electrons, and $\mu = e\hbar/(2m_e c)$ is the Bohr magneton. So kinetic energy/vol of two magnets at impact is $\sim (n\mu)^2$ (this is also of order $\int BdH$, the maximum stored energy, which reaches 1.8×10^6erg cm^{-3} for Sm-Co magnets, consistent with the order of mag estimate). Putting n of order factors times $1/a_0^3$, where $a_0 = \hbar^2/m_e e^2$ is the Bohr radius, and the binding energy/vol as of order 1Ry/Bohr radius cubed, where 1Ry$= e^4 m/\hbar^2$, we find kinetic/binding$= \alpha_f^2$, where $\alpha_f = e^2/(\hbar c) = 1/137$ is the fine structure constant. This is not far below typical yield stresses, so if the energy of contact isn't shared uniformly, they are likely to chip.

4. Magnetic fields maintained by the motion of conducting fluids (fluid dynamos) abound in nature. Dynamo theory is a well-developed branch of applied physics. However, the corresponding experimental subject does not exist. Can you explain why? Hint: The kinematic dynamo equation reads

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \kappa_M \nabla^2 \mathbf{B}$$

   By itself, the first term describes the motion of magnetic field lines frozen into perfectly conducting fluid. Kinetic energy is transferred into magnetic energy as fluid motions stretch field lines. This is the crux of dynamo action. The second term arises from ohmic dissipation. It describes the diffusion of magnetic field lines out of imperfectly conducting material. Form a dimensionless ratio known as the magnetic Reynolds number, R_M, which describes the relative importance of the dynamo to the dissipative term. Estimate the largest value of this number that one might achieve in a laboratory experiment on Earth.

¿From the magnetic induction equation we have:

$$\frac{\partial \mathbf{B}}{\partial t} = K_M \nabla^2 \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B})$$

The dimensional ratio between the terms on the r.h.s. gives

$$R_M = \frac{vL}{K_M}$$

where L is a typical lengthscale over which the magnetic field and/or velocity changes. R_M is called the magnetic Reynold's number.

Now, let's try to estimate it's magnitude. The highest value of K_M for highly conducting fluids you can hope to achieve is around $10^4 \mathrm{cm}^2\mathrm{s}^{-1}$ (molten iron). A typical velocity and length is, say 100 cms^{-1} and a typical length is $100cm$. This gives a Reynold's number of $R_M = 1$ which would mean that the field would ohmic decay too quickly. One would need an $R_M \sim 10 - 100$ to study dynamos.

However, in Tokamacs you can momentarily reach R_M $100 - 1000$, but there is still the question of how to study the dynamo effect in practice since it is a rather closed system.

5. At energies far below 100 GeV, weak interaction amplitudes depend on the Fermi constant $G_F = 1.4 \times 10^{-49}$ erg cm^3 [historical note not needed for the problem: Fermi worked out this problem in 1935, with great success; the Weinberg-Salam-Glashow theory in 1972 related G_F to the mass M_W of a hypothetical W particle by $G_F \simeq 4\pi(e\hbar/M_W c)^2$, also with great success.] Physical decay rates, cross-sections, etc., depend on the square of amplitudes, and hence are linear in G_F^2.

   a) The weak beta-decay rates of mirror nuclei (neutrons become protons and protons become neutrons) have dimensionless matrix elements of order unity. The decay rates ($1/\tau$, where τ is the half-life) depend on G_F, Planck's constant $\hbar$, the speed of light c, the energy E released in the decay, and the electron mass m_e. Use the Buckingham Pi theorem to identify all the independent dimensionless quantities.

   b) For $E \gg m_e c^2$, the weak decay rate does not depend on m_e. Use this fact, the information given in the statement of the problem, and the Pi theorem determine a formula for the half-life to beta decay, and use this formula to compute the neutron's half-life for beta-decay to a proton ($E = 0.8\,\mathrm{MeV}$ —the formula works well even for $E \sim m_e c^2$).

   c) The actual neutron half-life is 10 minutes. How big is the dimensionless number missing from your formula? Would it have helped if you had used h instead of $\hbar$ in your order-of-magnitude estimate?

At energies far below 100 GeV, weak interaction amplitudes depend on the Fermi constant $G_F = 1.4 \times 10^{-49}$ erg cm^3 [historical note not needed for the problem: Fermi worked out this problem in 1935, with great success; the Weinberg-Salam-Glashow theory in 1972 related G_F to the mass M_W of a hypothetical W particle by $G_F \simeq 4\pi(e\hbar/M_W c)^2$, also with great success.] Physical decay rates, cross-sections, etc., depend on the square of amplitudes, and hence are linear in G_F^2.

a) The weak beta-decay rates of mirror nuclei (neutrons become protons and protons become neutrons) have dimensionless matrix elements of order unity. The decay rates

($1/\tau$, where τ is the half-life) depend on G_F, Planck's constant $\hbar$, the speed of light c, the energy E released in the decay, and the electron mass m_e. Use the Buckingham Pi theorem to identify all the independent dimensionless quantities.

There are 6 independent dimensional quantites ($G_F, \hbar, c, E, m_e$ and τ) and 3 dimensions ($[M], [L], [T]$). By the Buckingham Pi Theorem, there are 6-3=3 independent dimensionless quantities. A simple set is

$$\Pi_1 = \frac{\hbar}{E\tau}$$

$$\Pi_2 = \frac{E}{m_e c^2}$$

and

$$\Pi_3 = \frac{G_F m_e^2 c}{\hbar^3}.$$

b) For $E \gg m_e c^2$, the weak decay rate does not depend on m_e. Use dimensional analysis to give a formula for the half-life to beta decay, and use this formula to compute the neutron's half-life for beta-decay to a proton ($E = 0.8\,\mathrm{MeV}$ —the formula works well even for $E \sim m_e c^2$).

Since $1/\tau \propto G_F^2$, independent of m_e, the only dimensionless choice is

$$\Pi_1 \sim \Pi_2^4 \Pi_3^2,$$

or

$$\frac{1}{\tau} \sim \frac{G_F^2 E^5}{\hbar^7 c^6} \sim 0.06\mathrm{s}^{-1}.$$

c) The actual neutron half-life is 10 minutes. How big is the dimensionless number missing from your formula? Would it have helped if you had used h instead of $\hbar$ in your order-of-magnitude estimate?

A factor of 40 would give the proper half life of 10 minutes. Since $\tau \propto \hbar^7$, letting $\hbar \to h$ makes τ increase by a factor 4×10^5, which is now too large. A lot of physics is hidden in those factors of 40 or 10^4 which our dimensional derivation does not elucidate. A version of the 'exact' equation in the old Fermi (pure vector coupling) theory can be found in Bethe & Morrison *Elementary Nuclear Theory*, p. 222 (note that this neglects coulomb modifications to the electron wave function, a good approximation for low Z nuclei like hydrogen, and consistent with this problem, where Z was not a parameter). In this theory, the RHS of our equation for $1/\tau$ should be multiplied by $1/(60\pi^3 \ln 2)$. At our 0.8 MeV energy, the approximation that $E \gg m_e c^2$ is not too good, and the 'exact' pure vector result is larger than the asymptotic large E result by a factor of 5.8. Putting these factors in, you will notice that we get $1/\tau = 1/(62\,\mathrm{min})$. However in 1957 Wu discovered that the weak interactions violate parity, and in 1958 Feynman and Gell-Mann proposed that the weak current had an axial-vector (non-parity conserving) component as well as the vector component of Fermi's old theory, the so-called $V - A$ theory. The axial-vector current contribution to the neutron decay rate is a factor of 4.9 times the vector contribution, so the correct lifetime is $62/(1+4.9) = 10$ min.