Ph103b: Solutions to Problem Set 1

Problem 1. *The "Planck mass" is defined as $(\frac{\hbar c}{G})^{\frac{1}{2}}$. As energy, what is this worth in gallons of gasoline?*

We use $\hbar c \simeq 200\,\mathrm{MeV\,fm}$, or using meters instead of femtometers, $\hbar c \simeq 2 \cdot 10^{-7}\,\mathrm{eV\,m}$. Then the Planck mass is

$$\sqrt{\frac{\hbar c}{G}} \sim \left(\frac{2 \cdot 10^{-7}\,\mathrm{eV\,m}}{6.7 \cdot 10^{-11}\,\mathrm{m^3\,s^{-2}\,kg^{-1}}} \times \frac{1.6 \cdot 10^{-19}\,\mathrm{J}}{1\,\mathrm{eV}}\right)^{1/2} = \sqrt{4 \cdot 10^{-16}\,\mathrm{kg}^2} = 2 \cdot 10^{-8}\,\mathrm{kg}. \tag{1.1}$$

Converting this mass to an energy,

$$E_{\mathrm{Planck}} \sim 2 \cdot 10^{-8}\,\mathrm{kg} \times c^2 \sim 2 \cdot 10^{-8}\,\mathrm{kg} \times 10^{17}\,\mathrm{m^2\,s^{-2}} \sim 2 \cdot 10^{9}\,\mathrm{J} \sim 5 \cdot 10^{5}\,\mathrm{kcal} \tag{1.2}$$

Gasoline is like a fat. From any nutrition label (e.g. on a can of coconut milk), fats provide $\sim 9\,\mathrm{kcal/g}$. So we'll take gasoline to be $\sim 10\,\mathrm{kcal/g}$ (Purcell's sheet also gives $10\,\mathrm{kcal/g}$). Converting the energy to liters of gasoline, and assuming that the density of gasoline is roughly that of water, $\rho \simeq 1\,\mathrm{g\,cm^{-3}}$,

$$E_{\mathrm{Planck}} \sim 5 \cdot 10^{5}\,\mathrm{kcal} \times \frac{1\,\mathrm{g}}{10\,\mathrm{kcal}} \times \frac{1\,\mathrm{cm}^3}{1\,\mathrm{g}} \times \frac{1\,\ell}{1000\,\mathrm{cm}^3} \sim \boxed{50\,\ell}, \tag{1.3}$$

which is about a full tank of gas.

Problem 2. *Cooking dishes are composed of special glasses that have low coefficients of expansion. For example, the linear coefficient of thermal expansion α, in units of inverse centigrade degree, is 1×10^{-5} for commercial glass, 3×10^{-6} for pyrex, and 8×10^{-7} for vycor. Why is ordinary glass inappropriate for cooking vessels? Be quantitative.*

Heating from $20\,^\circ\mathrm{C}$ to $250\,^\circ\mathrm{C}$ produces $\Delta T \sim 200\,^\circ\mathrm{C}$. For glass, $\alpha \sim 1 \cdot 10^{-5}\ ^\circ\mathrm{C}^{-1}$, so $\epsilon = \alpha \Delta T \sim 1 \cdot 10^{-3}$. The critical strain, ϵ_{crit} for most materials is $\sim 10^{-3}$–10^{-2}. For brittle materials like glass, we will take $\epsilon_{\mathrm{crit}} \sim 10^{-3}$ (which may be slightly optimistic). So regular glass would likely break in the oven. For Pyrex, where $\alpha_{\mathrm{pyrex}} \simeq 0.3\alpha_{\mathrm{glass}}$, the same temperature change would produce $\epsilon \simeq 0.3\epsilon_{\mathrm{crit}}$, which has a reasonable margin of safety. Though one of the TAs shattered a Pyrex oven dish putting it close to the flame under a gas broiler, where $\Delta T \simeq 350\,^\circ\mathrm{C}$, so maybe the safety margin isn't that much. Vycor would be very safe (and probably very expensive.)

Problem 3. *Benjamin Franklin noticed that a given amount of oil dropped on a lake's surface could not be induced to spread beyond a certain area. How much oil would be required to cover Millikan pond?*

Assuming the oil spreads into a monolayer, of thickness $d \sim 3\mathring{A}$, the total mass of oil is roughly $m \sim \rho A d$, where A is the area of Milikan pond. Taking the area to be $A \sim 15\,\mathrm{m} \times 60\,\mathrm{m} \sim 10^3\,\mathrm{m}^2 = 10^7\,\mathrm{cm}^2$, we get

$$m \sim 1\,\mathrm{g/\,cm^3} \times 10^7\,\mathrm{cm}^2 \times 3 \cdot 10^{-8}\,\mathrm{cm} \sim \boxed{1\,\mathrm{g}}. \tag{1.4}$$

As was pointed out by several students after class and in section, the oil may bead instead of spreading out (for example, olive oil floating in dressing). The minimum gravitational energy state of a drop of light liquid like oil on water is when the liquid is spread to as thin a layer as possible. However, this spreading can sometimes be prevented by surface tension. Consider the net inward force on a unit element of the circumference of the drop at the water surface: $\gamma_{oa} \cos\theta_a + \gamma_{ow} \cos\theta_w -$

γ_{wa} (here γ_{wa} is the surface tension of water in air, γ_{oa} is that for oil in air, and γ_{ow} is that for the oil-water interface). If $\gamma_{wa} < \gamma_{oa} + \gamma_{ow}$, there is an equilibrium for a drop with finite contact angles θ_a into the air and θ_w into the water. However, if the inequality is reversed, $\gamma_{wa} > \gamma_{oa} + \gamma_{ow}$, then there is no equilibrium even for a perfectly thin layer $\theta_w = \theta_a = 0$. The oil will then spread because it is energetically favorable to minimize the area of the water-air interface (and gravity also favors that solution). If spreading is favored, the spreading will continue until the oil forms a monolayer. Some examples of oils, which we hope to use in a demo, include hexane ($\gamma_{oa} \simeq 18\,\mathrm{erg\ cm^{-2}}$) and octane ($\gamma_{oa} \simeq 22\,\mathrm{erg\ cm^{-2}}$); both have $\gamma_{ow} \simeq 11\,\mathrm{erg\ cm^{-2}}$. Water has $\gamma_{wa} \simeq 70\,\mathrm{erg\ cm^{-2}}$, *so oils spread on clean water.* However if the container is dirty, so the water already has a thin film of oil on it, added oil may not satisfy the inequality required for spreading on the dirty surface, so may bead up in a drop, with the surface tensions providing the forces to pull it above water level against gravity. Lord Rayleigh in 1890 first used the technique of measuring the area of monomolecular films produced by known drop volumes to determine the sizes of molecules.

Problem 4. *Diffusion of perfume*

a) *Estimate the time required for perfume to diffuse across a room of size comparable to the one in which our class is held. Is this timescale in accord with your experience?*
b) *How does perfume usually spread in air?*

a) For the classroom, $R \sim 3000\,\mathrm{cm}$ (if you're going to square something, it's convenient to fudge the estimate to be either an even power of ten, or a factor of three times one). Perfume molecules are probably heavier than air molecules (mostly N_2), so instead of $\nu \sim 0.2\,\mathrm{cm^2/s}$, we'll use $\nu \sim 0.1\,\mathrm{cm^2/s}$ (also a convenient number), so

$$t_{\text{diffusion}} \sim R^2/\nu \sim \frac{10^7\,\mathrm{cm}^2}{0.1\,\mathrm{cm^2/s}} = 10^8\,\mathrm{s} \sim \boxed{3\,\mathrm{yr}}. \tag{1.5}$$

b) $\boxed{\text{Convection}}$—air currents carry the molecules around the room (drift beats than diffusion for large distances).

Problem 5. *My teacup is impervious to nucleons.*

a) *At absolute zero, how many neutrons can I put in the cup before my cup runneth over?*
b) *How many protons?*

a) The cup runneth over when the uncertainty in velocity of the degenerate neutrons is enough to escape the gravitational well (the height of the cup). If the number density of neutrons is n, then each neutron is 'confined' to a cube of side $\Delta x \sim n^{-1/3}$; from the uncertainty principle, $\Delta p \sim \hbar/\Delta x \sim \hbar n^{1/3}$, and $\Delta v \sim \hbar n^{1/3}/m_{\mathrm{n}}$. Confined is in quotes because the neutron wave functions actually extend over the whole cup. You solve for the energy levels in a three-dimensional box, and put neutrons in starting from the ground state, *i.e.*, from longer to shorter wavelengths (2 neutrons in each state, spin up and spin down, as allowed by the Pauli principle). The lowest energy neutrons will have wavefunctions with wavelengths comparable to the box size; the highest energy neutrons will have wavefunctions with wavelengths comparable to $n^{-1/3}$ (see any solid state text for the honest derivation of the Fermi energy or Fermi velocity). These highest energy neutrons will jump out of the cup first, as we increase the number density, and it is these whose energy we are estimating with our 'uncertainty velocity' method for finding the number density threshold.

When the velocity of the highest energy neutrons is comparable to the escape velocity, $v_{\mathrm{e}} \sim \sqrt{gL}$, where L is the side length of the cup, the neutrons will get out. So $\Delta v \sim \sqrt{gL}$. Substituting for Δv, we get

$$n^{1/3} \sim \frac{m_{\mathrm{n}}}{\hbar}\sqrt{gL}. \tag{1.6}$$

Now we use a useful trick, based on $\hbar c \simeq 200\,\mathrm{MeV\ fm}$. You can shift powers of ten from the electron-volts to the meters, to use the most convenient units for the problem; here we want centimeters because the cup has sides of a few centimeters, so we use $\hbar c \simeq 2 \cdot 10^{-5}\,\mathrm{eV\,cm}$. But there's no c, so we fix that, by multiplying by c/c, to get an $\hbar c$ in the denominator. Then we multiply by c/c again, to get $m_{\mathrm{n}}c^2$ (which you know is $\simeq 1\,\mathrm{GeV}$—this is another trick, to avoid looking up particle masses). We find

$$n^{1/3} \sim \frac{m_{\mathrm{n}}c^2}{\hbar c}\frac{\sqrt{gL}}{c}. \tag{1.7}$$

Putting in the numbers (taking $L \sim 10\,\mathrm{cm}$),

$$\begin{aligned} n^{1/3} &\sim \frac{10^9\,\mathrm{eV}}{2\cdot 10^{-5}\,\mathrm{eV\,cm}} \times \frac{\sqrt{1000\,\mathrm{cm/\,s^2} \times 10\,\mathrm{cm}}}{3\cdot 10^{10}\,\mathrm{cm/\,s}} \\ &\sim 2\cdot 10^5\,\mathrm{cm}^{-1}. \end{aligned} \tag{1.8}$$

With this number density, the number of neutrons in the cup is

$$nL^3 = (n^{1/3}L)^3 \sim \left(2\cdot 10^5\,\mathrm{cm}^{-1} \times 10\,\mathrm{cm}\right)^3 \sim \boxed{10^{19}}. \tag{1.9}$$

Note that if neutrons were bosons, then we could pack them all into the same state: they would each have wavefunctions with wavelength comparable to L and their uncertainty energy would be negligible.

b) Protons, like neutrons, obey Fermi statistics, but protons also have charge, and most likely the Coulomb repulsion will kick protons out before the Pauli repulsion will. If there are N protons spread around the cup, a single proton sees an electrostatic repulsive potential $U \sim Ne^2/L$. When $U \sim m_{\mathrm{p}}gL$, protons will jump out. So, $Ne^2 \sim m_{\mathrm{p}}gL^2$. To avoid remembering e in esu, or any other system, we use the $\hbar c$ trick again, because $e^2/\hbar c$ is defined to be the fine structure constant, which in this class is 0.01. So we divide both sides by $\hbar c$, and also multiply the right side by c^2/c^2 to get a $m_{\mathrm{p}}c^2$. These manipulations give

$$N\frac{e^2}{\hbar c} \equiv N\alpha \sim \frac{m_{\mathrm{p}}c^2}{\hbar c}g\left(\frac{L}{c}\right)^2. \tag{1.10}$$

Putting in numbers, and moving the α to the other side,

$$\begin{aligned} N &\sim \alpha^{-1}\frac{10^9\,\mathrm{eV}}{2\cdot 10^{-5}\,\mathrm{eV\,cm}} \times 1000\,\mathrm{cm/\,s^2} \times \left(\frac{10\,\mathrm{cm}}{3\cdot 10^{10}\,\mathrm{cm/\,s}}\right)^2 \\ &\sim 100 \times 5\cdot 10^{13}\,\mathrm{cm}^{-1} \times 1000\,\mathrm{cm/\,s^2} \times 10^{-19}\,\mathrm{s}^2 \sim \boxed{0.5}. \end{aligned} \tag{1.11}$$

So one proton is about the limit; putting in another will push the first out of the cup. Moral of the story: gravity is weak.