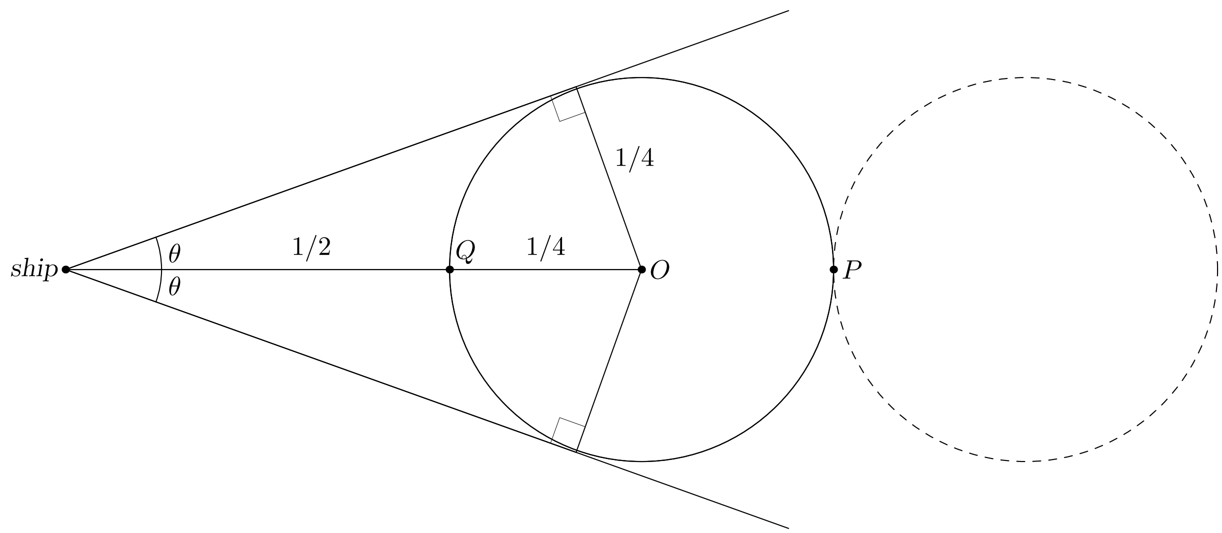
Solutions to Problem Set 6

1. Ship Wakes
   a) Prove that surface gravity waves generated by a ship in steady motion are confined to a wedge with opening angle $2\theta = 2\sin^{-1}(1/3)$.
   b) How does the wavelength of forward propagating waves depend upon the speed of the ship?

a) Let the ship move to the right, with velocity v. In the diagram below, the ship generates a backwards and a forwards wave when it is at P.



After some time t, the waves have grown. The solid circle shows the forward propagating wave, and the dotted circle shows the backwards one. We're interested only in the forward wave since it sets the opening angle of the wake; the backwards circle is quite a ways inside in the wake. The tangent lines mark the edge of the wake.

We will measure distance in units of vt, so that the length from P to the ship is 1. The group velocity of gravity waves on deep water is half the phase velocity. So the forward wave moves with half the phase speed, and the phase speed is the ship's velocity, v. So point Q, the farthest point in the forward wave, has advanced $1/2$ from P; therefore the circle has radius $1/4$. The right triangle has hypotenuse $3/4$ and height $1/4$, so $\theta = \sin^{-1}(1/3)$, and the opening angle is twice that, or $2\sin^{-1}(1/3)$.

b) The phase velocity, ω/k, is also the ship velocity, v. So $\omega = vk$. The dispersion relation for gravity waves on deep water is $\omega^2 = gk$. Substituting $\omega = vk$ into the dispersion relation gives $v^2k^2 = gk$ or $k = g/v^2$. So the wavelength is $2\pi/k = 2\pi v^2/g$. When $1/k$ is of the order of the ship length (so that $\mathrm{Fr} = v^2/gl \sim 1$), the ship will tilt, like one of those speedy cigarette-shaped motor boats you see in James Bond movies and Miami Vice.

A tilted hull means huge ram pressure drag which effectively limits the ship's velocity to $v = \sqrt{gl}$. For a 4 m motor boat, this speed is about 7 m/s. Another way to say this is that the Froude number is ~ 1 for the maximum ship speed (unless you have a really powerful engine). Fr ~ 1 also set the maximum walking speed for humans, cats and other sundry animals.

2. River Rapids

   Two conditions must be satisfied for the formation of rapids. The Froude number of the flow must exceed unity, and the river bottom must have boulders of size comparable to the river depth. Assume that the boulders are sparse so that their presence does not affect the mean flow.

   a) What is the limiting downstream slope required to produce rapids? Derive an analytic expression and then evaluate it numerically.

   a) The expression for river speed given in class was

$$v \sim \sqrt{\alpha g h} \frac{\ln h/z_0}{f}$$

where α is the slope, h is the height of the river, z_0 is the "roughness scale" at the bottom of the river, and f is the fudge factor relating the scale over which the velocity changes to the vertical height in the stream. The first criterion for rapids is that the Froude number $v^2/gh \geq 1$. Thus,

$$v^2 = gh\alpha \left(\frac{\ln h/z_0}{f}\right)^2 \geq gh$$

which implies

$$\alpha \geq \left(\frac{f}{\ln h/z_0}\right)^2 .$$

The second criterion for rapids is that the river depth is of the order of the size of the boulders in the stream. Thus, we take h to be ~ 1 m, our canonical boulder. Using $z_0 \sim 30$ cm and $f \sim 0.4$ as derived in class, we obtain $\alpha \geq 6°$.

3. Atmospheric Scintillation

   The atmosphere contains large cells of gas at very different temperatures separated by shear layers (one obvious place is near the ground, another is in convection cells). The shear is turbulent, so the turbulence mixes the hot and cold air down to very fine scales. The resulting complicated temperature distribution has a roughly Kolmogorov spectrum, such that two parcels of air separated by distance ℓ have root-mean-square temperature differences of order

$$\langle \delta T^2 / T^2 \rangle^{1/2} = 5 \times 10^{-5} (\ell / 1\,\mathrm{cm})^{1/3} .$$

   a) Explain why stars (angular size $< 10^{-7}$ radian) twinkle, but planets (angular size $\sim 5 \times 10^{-5}$ radian) don't.

   b) Estimate the timescale on which stars twinkle.

   Note: the index of refraction of air at optical wavelengths satisfies $n - 1 \approx 3 \times 10^{-4}$.

a) The Fresnel scale in the atmosphere is $\sqrt{\lambda D/2} \sim 5(\lambda/0.5\mu)^{1/2}\,\mathrm{cm}$, where $D \sim 10\,\mathrm{km}$ is the scale height of the atmosphere. Irregularities smaller than this blur the image of a point source by *diffraction*. Larger irregularities focus and displace the image by *refraction*.

Firstly we need to consider how far apart two light rays must be in order for them to be incoherent. When the phase difference, $\Delta\Phi$, between these two light rays is of order $\Delta\Phi \sim \pi$ then interference will be observed. Consider two paths of light separated by length l. The phase difference due to light crossing one blob of size l is

$$\Delta\phi = \delta n 2\pi \frac{l}{\lambda}$$

where δn is the difference in refractive index between neighboring blobs of length scale l. After passing through $N = D/l$ of these blobs (with randomly positive and negative δn) the phase difference will be

$$\Delta\Phi = N^{1/2}\Delta\phi = N^{1/2}\delta n \frac{2\pi l}{\lambda}.$$

Since $n-1$ is proportional to the density of air, at constant pressure $n - 1 \propto T^{-1}$, so $\delta n = (n-1)\delta T/T = (n-1)\; 5\times 10^{-5} l^{1/3}$ (l in cm), where on average $n - 1 = 3\times 10^{-4}$ for air in the visible. Therefore

$$\Delta\Phi = 2\pi D^{1/2} l^{5/6} 1.5\times 10^{-8}\lambda^{-1}.$$

The tilt of the wavefront on scale l is thus $\Delta\theta = \frac{\Delta\Phi}{2\pi}\frac{\lambda}{l} \sim 1.5\times 10^{-5} l^{-1/6}$. Thus the smallest scales (weakly) dominate the tilt. The Kolmogorov cascade continues down to $l \sim 0.1\,\mathrm{cm}$, where viscosity and diffusion smooth the fluctuations. Thus temperature fluctuations on scales of 0.1 cm–5 cm produce a fluctuating diffraction pattern (speckle pattern), whose size and position are modulated (by about $(0.1/5)^{1/6} \sim 50\%$) by the fluctuations on scales larger than the 5 cm Fresnel scale. This modulation is the cause of the observed twinkling. Detailed understanding of the twinkling in the earth's atmosphere is complicated by the fact that the Fresnel scale is comparable to the coherence length (at which $\Delta\Phi = \pi$, i.e., $l_c \sim 10(\lambda/0.5\mu)^{6/5}\mathrm{cm}$), by the fact that the wavefront tilt depends so weakly on l, so that a wide range of scales contributes, and by the fact that the atmosphere is not a thin screen (layers of the atmosphere within 100 m of the ground, at $\sim 3\,\mathrm{km}$ altitude, and at the base of the jet stream all contribute roughly equally to the scintillation).

b) The angular size of a star is much less than $\Delta\theta$ so that it will twinkle. However, planets subtend angles Θ much larger than $\Delta\theta$, so that although each $\Delta\theta$ "pixel" of the planet moves and changes in area (as easily observed through a telescope), since each "pixel" changes independently, the intensity of the planet changes only by a fractional amount of order one over the square root of the number of pixels, i.e. $\Delta\theta/\Theta \ll 1$. Thus planets don't twinkle much.

The timescale of twinkling is determined by the time it takes the winds in the atmosphere to convect a new set of temperature fluctuations into your line of sight to a star. Since the twinkling is dominated by fluctuations on about the coherence length $l_c \sim 10\,\mathrm{cm}$, the timescale is $t \sim l_c/v$. Taking a typical wind speed $v = 30\,\mathrm{mph} \sim 10^3\,\mathrm{cm\ s^{-1}}$, we find

$t \sim 10^{-2}$ s. To fix the speckle pattern, astronomers doing speckle interferometry must use exposure times much shorter than this. Your eye of course isn't sensitive to this most rapid twinkling: what you see is averaged over your eye's integration time (~ 0.05 s).

4. Southern California must find new supplies of fresh water. To be economically feasible the cost must be competitive with the current Pasadena price of 20 cents per 10^2 gallons. Use \$1.25 per gallon of gasoline as the cost of energy. In b) and c) below the total rate of supply enters into the unit cost. Assume that the supply is intended to meet the needs of 20 million people.

   a) **Towing icebergs from the arctic.** What size iceberg (assume it is a cube) would supply the needs of 20 million people for one year? How fast could we afford to tow it? Would melting in transit be a serious problem? At fixed speed, is it cheaper to tow one large iceberg or many smaller ones?

   b) **Pumping water over level ground from large rivers that empty into the Pacific Ocean near the border between Canada and the United States.** To keep the costs of pumped water comparable with that of current supplies, how large a diameter pipe would we have to build?

a) 20 m people using 100 gal/day gives about 10^{12} gal or $4 \times 10^{15}\,\mathrm{cm}^3$ of water used in a year. Since the density of ice is 0.9 that of water, this corresponds to an icecube $L = 1.5$ km on a side. Since 90% of the berg is underwater, the power required to tow it at speed v against the water drag ($Re \gg 1$) is $P_{tow} \sim (C_D/2)\rho_{\mathrm{H_2O}} v^3 L^2$, with $C_D \sim 1$. Thus the mechanical energy required for a trip from the arctic (time $t = D/v$, $D \sim 5000$ km) is $E_{\mathrm{tow}} \sim M_{berg} v^2 D/L$, where M_{berg} is the mass of the iceberg. Hence the (energy) cost per unit mass is $v^2 D/L$; so at fixed speed the water is cheaper if we tow one big iceberg instead of many small ones (drag means you pay for area, not mass!). The maximum economical towing speed is determined by the current price of water in energy units: 0.15 gallon of gas/100 gallon of water, or $U = 6 \times 10^8\,\mathrm{erg\ g^{-1}}$. Thus, with an engine of efficiency $\epsilon \sim 0.3$,

$$v_{\mathrm{max}} \sim \left(\frac{2}{C_D} \frac{L}{D} U \epsilon \right)^{1/2} \sim 500\,\mathrm{cm\ s^{-1}}\,,$$

or $17\,\mathrm{km\ h^{-1}}$. At this speed, the journey would take about 2 weeks. [Note that the engines required are amazing: $P_{\mathrm{tow}} \sim 200\ GW$, thus using a significant fraction of the total US energy-generating capacity. Large aircraft carriers are each powered by about 8 nuclear reactors, with total capacity of $\sim 200\ MW$ of shaft power. Towing the iceberg at v_{max} would thus require $\sim 10^3$ aircraft carriers! Only 10 carriers would be required to tow it at 1 m/s, though.] Melting: sunlight melts (80 cal/g) about a meter per month off the top. Heat conduction from surrounding sea water through the turbulent boundary layer melts ~ 20 meters per month of the bottom and sides. Hence melting isn't a problem.

b) Pump must carry 10^{12} gal/year, or $\dot{V} = 10^8\,\mathrm{cm^3 s^{-1}} = \pi r^2 v$, in pipe of radius r at flow speed v. Re is large, so the flow is turbulent, and the energy dissipated per unit volume in the pipe of length $D \sim 2000$ km is $\sim \rho v_t^2 v/r$, where $v_t \sim v/(2.4\ln(r v_t/\nu)) \sim v/30$ is the velocity of the turbulent eddies (ν is kinematic viscosity of water). Since it takes the water time D/v to traverse the pipe, the energy cost per gram of water is $v_t^2(D/r) = \dot{V}^2 D r^{-5}/(30\pi)^2$. To be economical, this must be less than the cost of energy U (calculated in (b) above) times the pump efficiency $\epsilon \sim 0.3$. Thus $r > 2\,\mathrm{m}$.

5. Water gets from the roots of trees to their leaves through the long hollow cells of the xylem. The radii of the xylem cells are $\sim 20\,\mu\mathrm{m}$, and the pores in the leaves have radii ~ 50Å.
   a) By arguments similar to those given in class for solids, estimate the tensile strength of pure water, with no dissolved gasses (the units are those of pressure, so the tensile strength is also called the cavitation pressure).
   b) One limit to the height of a tree is set by the capillary pressure defect of water in the pores (show that the capillary rise in the xylem is insignificant: water doesn't pour out of the stumps of cut trees). Another limit is set by the tensile strength of the column of water joining the root to the leaf pores. How close are the tallest trees to these limits?
   c) Estimate the velocity of flow of water up a tree trunk. What additional pressure drop (beyond the hydrostatic one) between the roots and the leaves are required to maintain this velocity of flow in the xylem?

a) The energy and mean separation is about 0.4eV/molecule (hydrogen binding) and 3 Ångström respectively. The bulk modulus defined as energy/separation^3 gives bulk modulus $2 \cdot 10^{10}$erg/cm^3, which is correct bulk modulus of water (AIP p 2-184). Typical tensile strength of pure materials is $10^{-2}B = 28$ erg/cm^3 $= 200$ atm. This is about the maximum tensile strength ever measured for pure, degassed water (AIP p. 2–204). 'Ordinary' measurements give about 10 atm, and seawater has about 0.5 atm (Denny, p. 255, Sterl's biology book).

b) From pressure balance, $\rho g h = 2 \cdot 10^8$, we get $h = 3$ km. The capillary pressure is $p = 2\gamma/R$ where γ is the surface tension, R is here the radius of the hollow cylinder. In our case R is the size of the pores in the leaves. So, from pressure balance we get $h = 2\gamma/(Rg\rho) = 3$km again. The capillary pressure for the xylem gives $h = 30$ cm. The tallest tree is 127 m, so about a factor of 20 below these limits. But surely the water has some dissolved gas. Real trees may be approaching real tensile strength limits?

c) ¿From Denny p. 72 : $\Delta p = 8\mu v/r^2 h$, where μ is the dynamic viscosity of water. Estimate v: if tree respires all rainfall under its leaf area, and leaf extent is 1/2 trunk height, then $v = 1\mathrm{m/yr}(\text{Area leaves}/\text{Area trunk}) = 1\mathrm{m/yr}(50\mathrm{m})^2/(1\mathrm{m})^2 = 10^{-2}\mathrm{cm/s}$. Real trees probably don't have water flowing up the whole trunk area, and probably need faster flow during the spring growing season when the rain falls. So 0.1cm/s seems reasonable, and is supposedly measured too! Using $\mu = 10^{-2}\mathrm{g/cm^2/s}$ for water, we get $\Delta p = 2\times 10^7\mathrm{dyn/cm^2} =$ 20atm for $h = 100$m. The hydrostatic drop $\rho g h$ for such a tree is 10 atm, so to keep the flow going one needs to triple the hydrostatic drop. Power for this comes from evaporation of water from leaves.