Ph103b: Solutions to Problem Set 8

Problem 1. *On one side of a 1-cm thick glass pot is a high-tech hotplate with an adjustable temperature. On the other side is water kept by boiling at a constant 373K. Below some hotplate temperature T_x most of the heat will be transferred through the glass by conduction, but above T_x the heat will mostly be carried by radiation. Estimate T_x. Hints: the absorption length for electromagnetic radiation in glass is $\gg 1\,\mathrm{cm}$ at $\lambda < 4\mu m$, and $\ll 1\,\mathrm{cm}$ at $\lambda > 5\mu m$. In water, the absorption length is $\gg 1\,\mathrm{cm}$ at $\lambda < 1\mu m$, and $\ll 1\,\mathrm{cm}$ at $\lambda > 2\mu m$.*

The conduction heat flux is

$$F_{\mathrm{c}} \sim K\frac{\Delta T}{\delta}, \tag{1.1}$$

where δ is the thickness of the glass, K is the thermal conductivity of glass, and ΔT is the temperature difference across the glass. Here $\Delta T = T_x - T_{\mathrm{water}}$.

If all the radiation got through the glass, and it all got absorbed in the water, then the radiative flux is

$$F_{\mathrm{r}} = \sigma T_x^4. \tag{1.2}$$

We ignore the reverse radiation flux from the boiling water back to the hotplate. Sunlight has a wavelength peak at $\lambda \sim 0.5\,\mu$ (green light), at a blackbody temperature $T \sim 6000\,\mathrm{K}$. So $T_{\mathrm{water}} \sim 400\,\mathrm{K}$ corresponds to $\lambda \sim 0.5\,\mu \times (6000/400) \sim 7\,\mu$, which can hardly travel in glass or water. We therefore ignore the water radiating back.

To find the critical temperature, we equate F_{c} and F_{r}:

$$K\frac{T_x - T_{\mathrm{water}}}{\delta} \sim \sigma T_x^4. \tag{1.3}$$

We solve for T_x by assuming that $T_x \gg T_{\mathrm{water}}$ (which we'll check at the end). Then

$$T_x \sim \left(\frac{K}{\sigma\delta}\right)^{1/3}. \tag{1.4}$$

Putting in $K \sim 10^{-2}\,\mathrm{cal\,cm^{-1}\,s^{-1}\,K^{-1}} = 4\cdot 10^5\,\mathrm{erg\,cm^{-1}\,s^{-1}\,K^{-1}}$ (from Purcell's sheet):

$$T_x \sim \left(\frac{4\cdot 10^5\,\mathrm{erg\,cm^{-1}\,s^{-1}\,K^{-1}}}{6\cdot 10^{-5}\,\mathrm{erg\,s^{-1}\,K^{-4}\,cm^{-2}} \times 1\,\mathrm{cm}}\right)^{1/3} \sim 1900\,\mathrm{K}. \tag{1.5}$$

As expected, $T_x \gg T_{\mathrm{water}}$.

But radiation from the hotplate can only heat the water if it has wavelength above $1.5\,\mu$—so the water will absorb the radiation—and below $4.5\,\mu$—so the radiation can get through the glass. A blackbody at $T_x \sim 1900\,\mathrm{K}$ has its power per wavelength peak at $\lambda \sim (6000\,\mathrm{K}/1900\,\mathrm{K}) \times 0.5\,\mu \sim 1.5\,\mu$, by the same argument used above to calculate the peak at 400 K. So let's say that only half of the Planck distribution gets absorbed; then we need to increase T_x by $\sim 2^{1/3} \sim 1.2$, so we'll take $\boxed{T_x \sim 2300\,\mathrm{K}}$.

Problem 2. *Estimate, using the principles of atomic absorption, the amount of dye needed to turn Millikan pond bright green. Typical dyes consist of two to five benzene rings with various small attachments to adjust the frequencies of electronic states and to make them soluble in the solvent of choice.*

Dyes absorb strongly: They use electric dipole transitions. (Quadrupole, magnetic dipole, and higher multipole transitions would be too weak to make a useful dye; you'd need gallons of the stuff.) Radiation excites an electron to a higher energy level; in the absorption band, instead of reradiating the energy, the electron dumps its energy into molecular vibrations and rotations, thereby generating heat instead of light.

In the harmonic oscillator model, the integrated cross-section is $\int \sigma(\nu)\, d\nu = \pi e^2/mc$. If the absorption band has width of $\Delta\nu$ (in frequency), then in the absorption band the cross-section is

$$\sigma_0 \sim \pi \frac{e^2}{mc} \frac{1}{\Delta\nu} = \pi r_0 \frac{c}{\Delta\nu} = \pi\alpha^2 a_0 \frac{c}{\Delta\nu}, \tag{1.6}$$

where we used the expression for the classical electron radius, $r_0 \equiv e^2/mc^2 = \alpha^2 a_0$ and ignored the oscillator strength adjustment.

A green dye transmits green light; since green is in the middle of the visible spectrum, the dye must have at least two absorption bands, to absorb the red end and the blue end. If the dye lets through say the middle third of the spectrum, and each absorption band absorbs another third, then we've accounted for the whole visible spectrum (of an octave). Thus, each absorption width is roughly a third of an octave. So we take $\Delta\nu \sim \nu_0/3$, and then1.6 becomes

$$\sigma_0 \sim \pi\alpha^2 a_0 \frac{c}{\nu_0/3} \sim \pi\alpha^2 a_0 3\lambda_0 = 3\pi\alpha^2 a_0 \lambda_0, \tag{1.7}$$

where λ_0 is the wavelength of the transition. We'll take $\lambda_0 \sim 5000\,\text{Å} = 5 \cdot 10^{-5}\,\text{cm}$ (an average of the two absorption band wavelengths). Then

$$\sigma_0 \sim 10 \times 0.5 \cdot 10^{-4} \times 0.5 \cdot 10^{-8}\,\text{cm} \times 5 \cdot 10^{-5}\,\text{cm} \sim 10^{-16}\,\text{cm}^2. \tag{1.8}$$

Thus if the vertical path through the pond is to have optical depth unity (an e-fold reduction in power at the red and blue ends), there must be 10^{16} dye atoms along each square centimetre. Millikan pond has area $A \sim 40\,\text{m} \times 15\,\text{m} = 600\,\text{m}^2 = 6 \times 10^6\,\text{cm}^2$, so the total number of dye molecules needed to color the pond is $10^{16} \times 6 \times 10^6 = 6 \times 10^{22}$. Common dye molecules contain one or two benzine rings with attachments, with typical molecular weight ~ 200. So the mass of dye needed is $\sim 200 \times m_p \times 6 \times 10^{22} \sim 20\,\text{g}$. If the dye has a density of $\sim 2\,\text{g cm}^{-3}$, that is $\boxed{10\,\text{cm}^3}$ or only 2 teaspoons of dye to vividly color the $\sim 10^8$ teaspoons of water in Millikan pond.

Problem 3. *The upper 3/4 of piano strings are bare steel wires, stretched to the yield point of steel.*

a) *Estimate the speed of transverse waves on such a piano string, and compare to the speed of sound in air.*

b) *Estimate (using only the properties of steel) the length of a piano string whose fundamental frequency ν_1 is middle C (262Hz).*

a) Transverse waves have velocity $c_t = \sqrt{T/\mu}$, where μ is the mass per unit length. This can be derived from the Pi theorem. The relevant variables are: the wave speed, c_t; the tension, T; the mass, M; and the length, L. Our string for now has zero thickness, so the only length scale is L. So we have four variables and three dimensions, therefore one Pi variable. A little fiddling gives $\Pi = c_t^2 M/TL$, so $c_t^{\sim} TL/M = T/\mu$, and $c_t \sim \sqrt{T/\mu}$. The magic constant here turns out to be unity.

Let A be the cross-section of string. The stress on the string is Y, the yield stress of steel, so the tension is $T \sim YA$. The mass per unit length is ρA, so $c_t = \sqrt{YA/\rho A} = \sqrt{Y/\rho}$. From the materials sheet, the yield stress for steel is $Y \sim 6 \cdot 10^9\,\mathrm{erg\,cm^{-3}}$, and $\rho \sim 8\,\mathrm{g\,cm^{-3}}$. So $\boxed{c_t \sim 3 \cdot 10^4\,\mathrm{cm\,s^{-1}}}$, which is about the speed of sound in air.

One may briefly wonder why c_t is not the same as the speed of sound in steel. The yield stress is defined as $Y = \epsilon B$, where B is the bulk modulus and ϵ is the yield strain. So $c_t = \epsilon^{1/2}\sqrt{B/\rho} = \epsilon^{1/2} c_s$, where c_s is the sound speed in steel. Since $\epsilon \sim 10^{-2}$, we find $c_t \sim c_s/10$. Compressional waves have a restoring force set by the interatomic forces, which are huge—so compressional waves move quickly. Transverse waves have a restoring force set by the tension. But you can't stretch most materials by anywhere near to the interatomic force—they flow or fracture long before that.

b) A piano string is fixed at both ends, so the lowest frequency (the fundamental) fits half a wavelength into the string length. Let λ be the wavelength and L be the string length. Then $\lambda = 2L$. Since $\lambda = c_t/f$, we have $L = c_t/2f \sim 3 \cdot 10^4/500 \sim \boxed{60\,\mathrm{cm}}$.

Problem 4.

During an orchestra concert, heat generated by the players, stage lights and the audience causes the temperature in the auditorium to rise by 5°K. Assuming the players take no corrective action,

a) *Estimate the fractional change in frequency of notes played by the wind instruments. Do the frequencies of their notes go up or down as the temperature rises?*
b) *Estimate the fractional change in frequency of notes played by the string instruments. Do the frequencies of their notes go up or down as the temperature rises? [hint: the coefficient of thermal expansion of spruce wood (used for piano and violin face and back plates) along the grain (which is parallel to the strings) is about 1/7 that of steel (used for strings).]*

a) The resonant frequency of a wind instrument of length L is given by $\nu = n(c_s/4L)$ if it has cylindrical bore (so the mouthpiece is a pressure maximum; clarinet, flute, trumpet, etc.) and by $\nu = n(c_s/2L)$ if it has conical bore (so the mouthpiece must also be nearly a node to avoid divergence of the pressure; oboe, bassoon, etc), with $n = 1, 2, 3$ for the various overtones blown. The temperature rise has two effects. First, it increases the speed of sound, and second, it lengthens the wind instrument:

$$\frac{\Delta\nu}{\nu} = \frac{\Delta c_s}{c_s} - \frac{\Delta L}{L} = \frac{1}{2}\frac{\Delta T}{T} - \alpha\Delta T\,,$$

where we have used $c_s = \sqrt{\gamma kT/m}$.

The fractional change in sound speed is $\frac{1}{2}\Delta T/T \sim 0.008$, which raises the frequency by this fraction. This change dominates the second term, the small frequency decrease from thermal expansion: from Purcell's sheet, the thermal expansion coefficient for solids, say brass, is $\alpha \sim 2 \cdot 10^{-5}/\mathrm{deg}$. Thus for a trumpet, the change in length due to thermal expansion lowers the frequency $\alpha\Delta T \sim 10^{-4}$, an order of magnitude less than the change in sound speed. In wind instruments the wood grains are parallel to the length of the instrument, so from part (b), the thermal expansion coefficient is 1/7 of that of metals, and the effect of the change in length on the frequency is two orders of magnitude less than that of the change in sound speed. Thus the wind instruments' frequencies $\boxed{\text{rise} \sim 0.8\%}$.

b) The fundamental frequency of a string of length L at tension t and mass per unit length μ is

$$\nu = \frac{1}{2L}\sqrt{\frac{t}{\mu}}\,. \tag{1.9}$$

The tension is given by

$$\frac{t}{\mu} = \frac{B}{\rho}\frac{L(T) - L_0(T)}{L_0(T)}\,, \tag{1.10}$$

where $L_0(T)$ is the length the relaxed string would have, B is the elastic modulus, and ρ the density. Neglecting the small changes in B and ρ, differentiating gives

$$\frac{\Delta(t/\mu)}{(t\mu)} = \frac{\Delta L/L - \Delta L_0/L_0}{(L - L_0)/L_0}\,. \tag{1.11}$$

Assuming the wood of the instrument body is thick enough that it is not significantly compressed by the string tension, the ends of the string will be forced to move with the wood: $\Delta L/L = \alpha(\text{wood})\Delta T$, while $\Delta L_0/L_0 = \alpha(\text{steel})\Delta T$.

Before the temperature rose, the strings were stretched to the yield point of steel, so $\epsilon = (L - L_0)/L_0 \sim 0.005$ is the strain. Thus we have

$$\frac{\Delta\nu}{\nu} = -\alpha(\text{wood})\Delta T + \frac{1}{2\epsilon}\big[\alpha(\text{wood})\Delta T - \alpha(\text{steel})\Delta T\big]\,. \tag{1.12}$$

The first term on the left (the effect of the change in string length on the frequency) is negligible compared to the later terms (the effect of the change in string tension), and since $\alpha(\text{wood}) = \alpha(\text{steel})/7$, we finally get $\frac{\Delta\nu}{\nu} = -(3/7)\alpha(\text{steel})\Delta T/\epsilon$. With $\alpha(\text{steel}) \sim 1.4\cdot 10^{-5}/\text{K}$ and the strain at yield $\epsilon \sim 0.005$, we find that the string instruments' frequencies $\boxed{\text{fall} \sim 0.6\%}$.

This is not a bad estimate: according to E. Lieber (1982), *On the Tuning Stability of Pianos*, *Das Musikinstrument* **31**, 602, the treble strings of a piano fall in pitch by $-1.65\,\text{cent/K}$. A cent is 1/100 of an equal-tempered semitone, i.e. a frequency ratio $2^{1/1200} = 1.00058$, so $\Delta T = 5\,\text{K}$ changes the frequencies of the treble piano strings by $\Delta\nu/\nu = 0.00058 \times (-1.65) \times 5 = -0.5\%$, quite close to our estimate. According to Lieber's measurements, the tenor strings of the piano (which are overwound with wire not under tension to increase μ and thus lower ν) drop in frequency by $-0.39\,\text{cent/K}$, or -0.1% for a 5 K rise in temperature. The piano thus gets noticeably out of tune with itself under such a temperature change (changes in humidity, which can swell the wood by 5%, are even worse), which is why concert pianos are tuned immediately before the concert.

Problem 5. *Boiling Water And Whistling Tea Kettles*

a) *It takes about 5 minutes to bring a liter of water to a boil on a kitchen stove.*
   - *i) How much power is being absorbed by the water?*
   - *ii) At what rate does the boiling water evaporate?*

b) *Many tea kettles come with whistles. The basic whistle is a hole of radius* $\approx 0.15\,\text{cm}$ *through which water vapor can exit the kettle.*
   - *i) At what velocity does water vapor exit the hole when water is boiling inside the kettle?*
   - *ii) What is the Reynolds number of the flow near the hole?*
   - *iii) Why does the kettle whistle and what determines its frequency?*

iv) Which multipole dominates the acoustic radiation? Estimate the acoustic power.

a) *i)* Say the water starts out at 20 °C. Raising the liter to to 100 °C takes

$$E_{\text{tot}} \sim 1000\,\text{g} \times 1\,\text{cal}\,\text{g}^{-1}\,{}^\circ\text{C}^{-1} \times 80\,{}^\circ\text{C} \sim 8 \cdot 10^4\,\text{cal} \sim 3 \cdot 10^{12}\,\text{erg}. \tag{1.13}$$

If this is dumped into the water in 5 minutes, the power is $P = E_{\text{tot}}/300\,\text{s} \sim 10^{10}\,\text{erg/s} = \boxed{1\,\text{kW}}$.

ii) The water evaporates at a rate $R = P/L_{\text{vap}}$. From Purcell's sheet, $L_{\text{vap}} \sim 10^4\,\text{cal/mol} \sim 500\,\text{cal/g}$, which is $L_{\text{vap}} \sim 2 \cdot 10^{10}\,\text{erg/g}$. So $R \sim 10^{10}/2 \cdot 10^{10} = \boxed{0.5\,\text{g/s}}$.

b) *i)* At STP, one mole of ideal gas vapor is 22.4ℓ. At 100 °C a mole has more volume, by a factor of 1.3, so we'll take $30\,\ell/\text{mol}$ as the conversion. At 18 g/mol, our 0.5 g of water is 0.025 mol, or $0.7\ell = 700\,\text{cm}^3$. The flux, F, is therefore $700\,\text{cm}^3\,\text{s}^{-1}$, and this is vA, where $A = \pi r^2$ is the cross-sectional area of the whistle hole, and v is the exit velocity. So

$$v \sim \frac{F}{A} = \frac{700}{\pi \times 0.15^2} \sim 10^4\,\text{cm}\,\text{s}^{-1} \tag{1.14}$$

ii) The Reynolds number is $\text{Re} \sim rv/\nu$, where ν is the viscosity of steam, which we take as approximately that of air, $\nu \sim 0.2$. Then $\text{Re} \sim 0.15 \times 10^4/0.2 \sim \boxed{10^4}$. The flow is turbulent.

iii) The turbulent flow at the hole oscillates back and forth with velocity v, shedding vortices from side to side, in the famous von Karman vortex pattern. The angular frequency of oscillation is roughly the time to cross the whistle hole, so $\omega \sim v/2r$, and $f \sim v/4\pi r \sim 10^4/1.5 \sim \boxed{7\,\text{kHz}}$.

iv) The whistle is an acoustic monopole (there is mass flux causing a volume change). From class,

$$P_{\text{monopole}} = 4\pi\omega^2 p_0 \frac{r^4 v^2}{c_{\text{s}}^3}. \tag{1.15}$$

Since $\omega \sim v/2r$ and $p_0 \sim \rho c_{\text{s}}^2$,

$$P_{\text{monopole}} \sim \pi\rho r^2 v^3 \frac{v}{c_{\text{s}}}. \tag{1.16}$$

For the hot steam, $\rho = 1\,\text{mol}/30\ell \sim 6 \cdot 10^{-4}\,\text{g}\,\text{cm}^{-3}$, and $v \sim 10^4\,\text{cm}\,\text{s}^{-1}$. So with $v/c_{\text{s}} \sim 0.3$, we get

$$P \sim 3 \times 6 \cdot 10^{-4}\,\text{g}\,\text{cm}^{-3} \times (0.15\,\text{cm})^2 \times 10^{12}\,\text{cm}^3 \times 0.3 \sim 2 \cdot 10^7\,\text{erg} = 2\,\text{W}. \tag{1.17}$$

If the sound is emitted over a hemisphere, instead of a full sphere as for the case worked out in class, then $\boxed{P \sim 1\,\text{W}}$.

At a distance of 1 m, the intensity is $\sim 1/4\pi \sim 0.1\,\text{W/m}^2$, which is about 110 dB ($1\,\text{W/m}^2$ is 120 dB). The pain threshold is 120 dB, so 110 dB seems about right, or perhaps 5 dB too high.