Ph103b: Solutions to Problem Set 6

Note: we refer frequently to the results on falling balls, which can be found at `http://dope/oom/` under 'Bouncing Balls'.

Problem 1. *A steel ball and a solid ball of soft rubber (one which you can deform perceptibly by squeezing hard) are dropped onto a hard steel plate. What is the maximum height from which the steel ball can be dropped before its coefficient of restitution begins to drop due to inelastic deformations? For the rubber ball? Do the sizes of the balls matter?*

From the class on bouncing balls, we know the strain is

$$\epsilon \sim \left(\frac{\rho v^2}{\mathcal{M}}\right)^{1/5} \sim \left(\frac{\rho g h}{\mathcal{M}}\right)^{1/5}. \tag{1.1}$$

Taking the yield strain for steel to be $\epsilon \sim 0.005$ (as in class), and $\mathcal{M} \sim 2 \cdot 10^{12}\,\mathrm{erg\,cm^{-3}}$, we get $\rho g h \sim 6\,\mathrm{erg\,cm^{-3}}$. Taking $\rho \sim 8\,\mathrm{g\,cm^{-3}}$, we get $h \sim 0.001\,\mathrm{cm} \sim 10\,\mu$. That seems kind of small.

For rubber, we will find below that $\mathcal{M} \sim 10^7\,\mathrm{erg\,cm^{-3}}$ and $\rho \sim 1\,\mathrm{g\,cm^{-3}}$. Alternatively, we can estimate $\mathcal{M}$ by guessing that a finger can exert 10 'lb' of force over say $1\,\mathrm{cm}^2$, and this may deform the ball by perhaps 10% ($\epsilon \sim 0.1$). This force produces a stress $\sigma \sim 5000\,\mathrm{g} \times 10^3\,\mathrm{cm\,s^{-2}}/1\,\mathrm{cm}^2 \sim 5 \cdot 10^6\,\mathrm{erg\,cm^{-3}}$. Since $\sigma = \epsilon\mathcal{M}$, we get an estimate $\mathcal{M} \sim 5 \cdot 10^7\,\mathrm{erg\,cm^{-3}}$. The yield strain in soft rubber is ~ 1, so using $\mathcal{M} \sim 10^7\,\mathrm{erg\,cm^{-3}}$, we get $\rho g h \sim 10^7\,\mathrm{erg\,cm^{-3}}$; therefore $h \sim 10^4\,\mathrm{cm}$.

These heights can't be taken too literally because of the high exponent involved (a fifth power): a small—by the standards of this class—error in the required ϵ, or in the formula for ϵ, may make one or two orders of magnitude error in estimating $\mathcal{M}$ or h.

The size of the ball doesn't matter, only the height through which the center of mass falls.

Problem 2. *The blue superball which was dropped off the top of Millikan library has a diameter of 4.2 cm and a mass of 38 g. Resting on a hard surface, the radius of the circle of contact is measured to be* 0.13 cm.

a) *What is the elastic modulus of the rubber?*

b) *When you drop the rubber ball, it bounces with decreasing height and increasing frequency until it stops bouncing and begins simply to vibrate. What is the minimum time between bounces of the rubber ball?*

a) Resting on the ground deforms the ball. For the ball-drop discussed in class, the strain is $\epsilon \sim r/R$, which produces a stress $\sigma \sim \mathcal{M}\epsilon \sim \mathcal{M}r/R$, and it occurs over an area $\sim r^2$. So the restoring force is $F \sim \sigma r^2 \sim \mathcal{M}r^3/R$. This force must balance the weight, so $F = mg \sim \rho R^3 g$, and we get $\mathcal{M}r^3/R \sim \rho R^3 g$. Solving for the elastic modulus, we find

$$\mathcal{M} \sim \rho g R \left(\frac{R}{r}\right)^3. \tag{1.2}$$

With $R = 2.1\,\mathrm{cm}$ and $m = 38\,\mathrm{g}$, we calculate $\rho \approx 1\,\mathrm{g\,cm^{-3}}$. Putting these numbers into1.2:

$$\mathcal{M} \sim 1\,\mathrm{g\,cm^{-3}} \times 10^3\,\mathrm{cm\,s^{-2}} \times 2.1\,\mathrm{cm} \times \left(\frac{2.1\,\mathrm{cm}}{0.13\,\mathrm{cm}}\right)^3 \sim 10^7\,\mathrm{erg\,cm^{-3}}. \tag{1.3}$$

Typical values for soft rubber are $\mathcal{M} \sim 2 \cdot 10^7\,\mathrm{erg\,cm^{-3}}$.

b) When the contact time exceeds the time it takes the ball to fall the height its center of mass rises after a bounce, the ball will stop bouncing. From the notes, the contact time is

$$\Delta t \sim \frac{R}{v}\left(\frac{v}{c_\mathrm{s}}\right)^{4/5}, \tag{1.4}$$

where $c_\mathrm{s} \sim \sqrt{\mathcal{M}/\rho}$ is the sound speed in the ball. The fall time is $\sim v/g$. Equating these times and solving for the critical v, we find

$$v \sim \frac{(Rg)^{5/6}}{c_\mathrm{s}^{4/5}}. \tag{1.5}$$

The corresponding fall time is the inter-bounce time at the transition to vibration:

$$\Delta t \sim \frac{v}{g} \sim \left(\frac{R}{c_\mathrm{s}}\right)^{2/3}\left(\frac{R}{g}\right)^{1/6}. \tag{1.6}$$

The first factor in parenthesis, R/c_s, is the time for sound to cross the ball; the second factor contains $\sqrt{R/g}$, which is the time to fall a distance equal to the radius of the ball. The critical time is a weighted geometric mean of these two times.

Now let's put in numbers. Using1.3 for $\mathcal{M}$, we find that the sound speed is $c_\mathrm{s} \sim \left(10^7/1\right)^{1/2}\,\mathrm{cm\,s^{-1}} \sim 3000\,\mathrm{cm\,s^{-1}}$. Then

$$\Delta t_\mathrm{crit} \sim \left(\frac{2\,\mathrm{cm}}{3000\,\mathrm{cm\,s^{-1}}}\right)^{2/3} \times \left(\frac{2\,\mathrm{cm}}{1000\,\mathrm{cm\,s^{-2}}}\right)^{1/6} \sim 3\,\mathrm{ms}. \tag{1.7}$$

As the ball bounce time decreases, we should hear a steadily increasing frequency until around $f \sim (3\,\mathrm{ms})^{-1} \sim 300\,\mathrm{Hz}$, the sound should stop.

Problem 3. *Yet more fun with balls*

a) *If you drop the blue superball of problem 2 from height $h = 1\,\mathrm{m}$ with no spin while you are at rest, it will rebound with no spin. But if you drop it while walking, it will be spinning when it rebounds. Why? Is Galilean invariance wrong? Estimate the rotation frequency of the ball after bouncing.*

b) *You repeat the experiment, but now dropping the ball out of a car with horizontal velocity v_x adjustable to higher values than walking speed. Estimate the critical v_x above which the ball skids, in the speedometer units of your native land.*

a) During contact with the ground, the frictional force exerts a torque on the ball, which will give it a spin. Galilean invariance is not wrong; it just doesn't apply here because we have a preferred frame: the frame of the ground. If the ball has a horizontal velocity with respect to this preferred frame, it will rebound with spin (which is a frame-independent quantity).

If the ball doesn't slip, and has forward velocity v'_x after the bounce, then the angular velocity ω must be enough to make $\omega R = v'_x$. Conserving the forward kinetic energy (which gets split into rotational and and translational pieces), we get

$$\frac{1}{2}mv_x^2 = \frac{1}{2}mv_x'^2 + \frac{1}{2}\frac{2}{5}mR^2\omega^2 = \frac{7}{10}mv_x'^2. \tag{1.8}$$

Therefore, $v_x = \sqrt{5/7}\,v_x \sim 0.85 v_x$ and $\omega = 0.85 v_x/R$. Walking speed is $\sim 3\,\mathrm{mph}$ or $150\,\mathrm{cm\,s^{-1}}$. With $R = 2.1\,\mathrm{cm}$, we get

$$\omega \sim 60\,\mathrm{rad/s} \qquad \Rightarrow \qquad f \sim 10\,\mathrm{revolutions/second}. \tag{1.9}$$

b) Let c_f be the coefficient of friction of the ball on the surface. Then the maximum force along the surface is $F_x^{\max} = c_\mathrm{f} F_y$, where F_y is the downwards force (the contact force calculated in class). The angular momentum given to the ball in the contact time is $L \leq c_\mathrm{f}(F_y \Delta t)R$. But from part (a), we know that the linear momentum impulse is $F\Delta t \sim m v_y$, where v_y is the downwards velocity. So

$$L \leq c_\mathrm{f} m v_y R. \tag{1.10}$$

We also have $L = I\omega = (2/5)mR^2\omega$. From part (a), for no skidding, we know that $\omega \sim v_x R$ (taking $0.85 \sim 1$). Then we get $L \sim (2/5) m v_x R$. By combining this expression with1.10, we find the critical velocity:

$$v_x \leq \frac{5}{2} c_\mathrm{f} v_y. \tag{1.11}$$

In other words, for no skidding the angle of impact can't exceed $\tan^{-1}(5c_\mathrm{f}/2)$.

For $h \sim 100\,\mathrm{cm}$, the downwards velocity is $v_y = \sqrt{2gv_y} \sim 400\,\mathrm{cm\,s^{-1}}$. Taking $c_\mathrm{f} \sim 1$ gives $v_x \leq 1000\,\mathrm{cm\,s^{-1}}$, or 20 mph. A good runner could make the ball skid. Or just throw it (with no spin!).

Problem 4. *Mars has atmospheric pressure 5×10^{-3} earth atm, and radius 0.5 earth radii.*
a) Equipped with an oxygen mask, could an earth native bird fly on Mars?
b) Could it land?
c) Could a mouse on Mars hear a falcon coming towards it in a free-fall dive?

a) From the lecture on flight, we know that the minimum power to fly scales as $P_{\min} \sim g^{3/2}\rho_\mathrm{a}^{-1/2}$. Mars has slightly slightly less dense rock than the Earth (~ 0.8 times that of Earth). We can then find a scaling for the gravitational constant: $g = GM/R^2 \sim \rho G R^3/R^2 \propto \rho R$. So $g_\mathrm{mars} \sim 0.4 g_\mathrm{earth}$. The ideal gas law says $P \propto nT$, so $\rho \propto mP/T$. The temperature on Mars is roughly $-50\,{}^\circ\mathrm{C} \sim 220\,\mathrm{K} \sim 0.7 T_\mathrm{earth}$, and the atmosphere is CO_2, which has molecular mass ~ 1.5 times that of N_2. We are given that P is 0.005 times that of earth. Combining all these factors together, we get that the atmospheric density is $\rho \sim 0.01\rho_\mathrm{earth}$. The minimum power therefore gets scaled from its earth value by $0.4^{3/2} \times 0.01^{-1/2} \sim 2.5$. The minimum power is pretty high. Possibly a tough bird could manage it, but not for very long distances.

b) But the minimum-power speed scales as $(g/\rho_\mathrm{a})^{1/2}$, which means a factor of 6 increase in speed. The kinetic energy therefore increases by a factor of 36: the bird would surely break all its bones upon landing. And to land at a lower, survivable speed would take too much power.

c) Can the falcon can dive supersonically? The sound speed scales as the thermal speed, $\propto \sqrt{T/m}$. Using the scalings from part (a), we get the scaling for the sound speed as 0.7, so $c_\mathrm{s} \sim 230\,\mathrm{m\,s^{-1}}$. The terminal velocity of the falcon is given by the high-Re formula with $c_\mathrm{d} \sim 1$:

$$v \sim \left(\frac{mg}{0.5\rho A}\right)^{1/2}. \tag{1.12}$$

The drag coefficient is effectively unity at these transsonic speeds: the drag comes from the ram pressure, $\sim \rho v^2/2$, exerted over the frontal cross-section. Putting in $m \sim 1\,\mathrm{kg}$ and $A \sim 100\,\mathrm{cm}^2$, we get:

$$v \sim \left(\frac{1000\,\mathrm{g} \times 400\,\mathrm{cm\,s}^{-2}}{0.5 \times 10^{-5}\,\mathrm{g\,cm}^{-3} \times 100\,\mathrm{cm}^2}\right)^{1/2} \sim 2.8 \cdot 10^4\,\mathrm{cm\,s}^{-1}. \tag{1.13}$$

It's close, but probably the falcon can dive supersonically (which means the mouse won't hear its invitation to dinner before the meal is served.) And maybe the falcon shakes its feathers off flying so close to the sound barrier. But that's the least of its problems: The falcon will eat only one meal, if it's lucky (none otherwise)—the landing will most likely break all its bones. As an exercise to the reader, estimate how far into the ground the falcon embeds itself.

Problem 5. *How fast could a plant grow (height per unit time)? Consider limitations due to availability of light, water and* CO_2*.*

We will consider grass as our typical plant.

If there is no limit to the supply of water, ground based nutrients and carbon dioxide then the limiting factor is the availability of sunlight. We quote the average solar flux from the weather lecture: $S \sim 2 \cdot 10^5\,\mathrm{erg\,cm}^{-2}\,\mathrm{s}^{-1}$.

Burning plant material (carbohydrate) releases about $h \sim 4\,\mathrm{kcal/gm}$ or $2 \cdot 10^{11}\,\mathrm{erg/gm}$. Burning converts cellulose plus oxygen to CO_2 and H_2O. When the plant grows, it must therefore use at least an equal amount of energy from sunlight to drive the inverse reaction (H_2O and CO_2 converted into sugars and cellulose). Hence the growth rate of grass, if it uses sunlight with efficiency ϵ is

$$\frac{dm}{dt\,dA} = \frac{\epsilon S}{h} = 0.09\epsilon\,\mathrm{g\,cm}^{-2}\,\mathrm{day}^{-1} = 30\epsilon\ \mathrm{g\,cm}^{-2}\,\mathrm{year}^{-1}. \tag{1.14}$$

This rate tells how many tons of grass clippings you have to haul away. To estimate the rate at which the grass lengthens, we need to know the mean density of a grass patch. Grass blades have density $\rho_{\mathrm{grass}} \sim 1\,\mathrm{g\,cm}^{-3}$, but they fill only $f \sim 10^{-2}$ of the volume of a lawn (the evolutionary reason can be understood by noting that the grass blades in front of Millikan have $twh = 0.02\,\mathrm{cm} \times 0.2\,\mathrm{cm} \times 10\,\mathrm{cm}$, so a volume $4 \cdot 10^{-2}\,\mathrm{cm}^3$. But, averaged over a day, each blade shadows a mean volume $\simeq (1/2)h^2(1/3)w = 3\,\mathrm{cm}^3$ where no other grass can get sunlight. Hence the filling factor should be $f \sim 4 \cdot 10^{-2}\,\mathrm{cm}^3/3\,\mathrm{cm}^3 \sim 10^{-2}$, as observed). The growth rate, dh/dt, is thus

$$\frac{dh}{dt} \sim \frac{dm}{dt\,dA}\frac{1}{f\rho_{\mathrm{grass}}} \sim 2\,\mathrm{cm/day}\left(\frac{\epsilon}{0.2}\right)\left(\frac{0.01}{f}\right). \tag{1.15}$$

Since grass has to grow roots as well as blades, and the roots have mass at least comparable to the blades', probably less than half of this dh/dt will appear as above-ground growth. If chloroplasts get light of the optimum wavelength, they can convert $\sim 30\%$ to energy. But sunlight contains many photosynthetically useless photons. So we will take $\epsilon \sim 0.1$. Then $dh/dt \sim 0.5\,\mathrm{cm/day}$, about what is observed in a well-watered Pasadena lawn in summer.

What about water? A typical long carbohydrate is $(CH_2O)^n$, so about five percent of the plant's mass is hydrogen. Hydrogen must come from the water, and water is 10 percent hydrogen by mass. Hence to grow a gram of plant material requires $\sim 0.5\,\mathrm{g}$ of water. In recent years, Southern California's precipitation has been only $\sim 35\,\mathrm{cm/year}$, allowing $70\beta f^{-1}\,\mathrm{cm/year}$ or $0.2\beta f^{-1}\,\mathrm{cm/day}$ of plant growth, where β is the efficiency of absorbing and converting water to carbohydrate, and $f \sim 0.01$ is the fill factor used above. Taking $\beta \sim 0.1$ to account for losses in transpiration and evaporation, we find $dh/dt \sim 2\,\mathrm{cm/day}$. Comparing this rate to the sunlight-limited rate of $0.5\,\mathrm{cm/day}$, we see that it's not the meager total rainfall that limits the growth of Los Angeles grass. Rather, it's the variance: the long dry summer will kill all the grass before it can make it to winter and drink up. In fact grass and trees, like humans, lose most of their water through transpiration and evaporation. Desert plants like cacti have evolved to lose much less.

What about carbon dioxide? It must diffuse across a viscous boundary layer next to the grass; the boundary layer thickness, δ, will limit the mass flux:

$$F \sim D\frac{\Delta\rho}{\delta}, \tag{1.16}$$

where D is the diffusivity of CO_2 in air and $\Delta\rho$ is the density difference across the boundary layer. Since CO_2 is slightly more massive than N_2, we will take D to be $\sim 0.1\,\mathrm{cm^2\,s^{-1}}$. The carbon dioxide concentration in the atmosphere (mass/mass) is $\sim 4\cdot 10^{-4}$, so we will take $\Delta\rho \sim 4\cdot 10^{-7}\,\mathrm{g\,cm^{-3}}$. Now we only need δ. We will guess the wind speed near the ground is $v \sim 40\,\mathrm{cm\,s^{-1}}$. [There was some debate whether to use $v \sim 10\,\mathrm{cm\,s^{-1}}$ or a larger value such as $v \sim 40\,\mathrm{cm\,s^{-1}}$. So Sterl had the good idea to actually substitute some experiment for theory and noted the following:

A $10\,\mathrm{cm}$ grass blade bends about $45°$ in a strong breeze. The blades of grass in front of Millikan are approximately $0.2\,\mathrm{cm} \times 0.01\,\mathrm{cm} \times 8\,\mathrm{cm}$. Before they wilted, $1\,\mathrm{cm}$ could lift a dollar bill ($1\,\mathrm{g}$), deflecting by $1\,\mathrm{cm}$. Thus $8\,\mathrm{cm}$ should lift $m \sim (1/8)^2\,\mathrm{g}$, deflecting $8\,\mathrm{cm}$ (using the strut scaling from class) and requiring a force $mg \sim 1000/8^2\,\mathrm{dyne} \sim 20\,\mathrm{dyne}$. The free oscillation frequency of the blade is about $5\,\mathrm{Hz}$, which gives a force of $40\,\mathrm{dyne}$ for an $8\,\mathrm{cm}$ deflection, consistent with the dollar bill result.

Thus the wind speed at grass level required to bend the grass is

$$F = 20\text{–}40\,\mathrm{dyne} = \frac{1}{2} \times 0.2\,\mathrm{cm} \times 8\,\mathrm{cm} \times \rho_{\mathrm{a}} v^2, \tag{1.17}$$

giving $v \sim 200\,\mathrm{cm\,s^{-1}}$. Such uniform bending requires a strong breeze, but it probably indicates that the speeds at grass level are more than $10\,\mathrm{cm\,s^{-1}}$ on a normally breezy day. So we'll use $v \sim 40\,\mathrm{cm\,s^{-1}}$.]

Then

$$\delta \sim \sqrt{\frac{\nu w}{v}} \sim \sqrt{\frac{0.2\,\mathrm{cm^2\,s^{-1}} \times 0.2\,\mathrm{cm}}{40\,\mathrm{cm\,s^{-1}}}} \sim 0.03\,\mathrm{cm}, \tag{1.18}$$

where we used the width of the grass blades as the characteristic length. The flux of carbon dioxide is then

$$F \sim 0.1\,\mathrm{cm^2\,s^{-1}} \times \frac{4\cdot 10^{-7}\,\mathrm{g\,cm^{-3}}}{0.03\,\mathrm{cm}} \sim 10^{-6}\,\mathrm{g\,cm^{-2}\,s^{-1}}. \tag{1.19}$$

If the blade collects CO_2 over an area $wh \sim 2\,\mathrm{cm^2}$ with an efficiency α, then the mass rate is

$$\dot{M} \sim 2\,\mathrm{cm^2} \times \alpha F \sim 2\cdot 10^{-6}\alpha\,\mathrm{g\,s^{-1}} \sim 0.2\,\alpha\,\mathrm{g/day}. \tag{1.20}$$

The carbon in plant matter—in CH_2O—comes from CO_2, so doing the stoichiometry, 1 gram of CO_2 will make $\sim 1.5\,\mathrm{g}$ of plant matter. And $1\,\mathrm{g} \approx 1\,\mathrm{cm}^3$ of plant matter is a height $1\,\mathrm{cm}^3/tw \sim 250\,\mathrm{cm}$. Using1.20, we find that CO_2 limits the growth to $80\alpha\,\mathrm{cm/day}$. The uptake of CO_2 isn't perfect; maybe $\alpha \sim 0.1$, so the CO_2-limited growth rate is $\sim 8\,\mathrm{cm/day}$, indicating that grass, and maybe other plants, are not often CO_2-limited, but may be sunlight-limited, especially in winter.