Ph103b: Solutions to Problem Set 3

Problem 1. *Following a winter storm the interval between waves at Southern California beaches declined from* $17-19$ s *on Sunday, to* $16-18$ s *on Monday, and to* $15-16$ s *on Tuesday. Typical values are* $10-11$ s.
a) What was the maximum sustained wind speed during the storm?
b) How distant was the storm from Southern California?
c) How long ago did it take place?
d) What are upper limits on the size and duration of the storm?

a) The wind generates waves with $v_{\rm ph} \simeq v_{\rm wind}$. These waves are gravity waves on deep water, whose dispersion relation is $\omega^2 = gk$. Therefore the phase velocity is $v_{\rm ph} \equiv \omega/k = \sqrt{g/k}$. The group velocity is

$$v_{\rm g} \equiv \frac{\partial\omega}{\partial k} = \frac{1}{2}v_{\rm ph} = \frac{1}{2}\sqrt{\frac{g}{k}}. \tag{1.1}$$

Putting in $k = \omega^2/g$, we find $v_{\rm g} = g/2\omega$. In terms of the period, $T = 2\pi/\omega$,

$$v_{\rm g} = \frac{gT}{4\pi}. \tag{1.2}$$

The largest average period (on Sunday), when $T \simeq 18$ s, gives the largest group velocity (and therefore the largest phase velocity and wind speed):

$$v_{\rm g}^{\rm max} \simeq \frac{1000\,{\rm cm\,s^{-2}} \times 18\,{\rm s}}{4\times 3} \sim 1500\,{\rm cm\,s^{-1}}. \tag{1.3}$$

The maximum sustained wind speed, is $v_{\rm ph} = 2v_{\rm g}^{\rm max} = \boxed{3000\,{\rm cm\,s^{-1}}}$ or about $\boxed{100\,{\rm kph}}$.

b&c) As we saw in1.2, longer period waves move faster—which is why $T_{\rm Sunday} > T_{\rm Tuesday}$. On average, $T = 15.5$ s on Tuesday, and these waves lagged by two days; so when Sunday's waves hit, Tuesday's lagged a distance

$$L = 2\,{\rm days} \times v_{\rm g} \sim 1.7\cdot 10^6\,{\rm s} \times 1300\,{\rm cm\,s^{-1}} \sim 2.2\cdot 10^8\,{\rm cm}, \tag{1.4}$$

where $v_{\rm g} \sim 1300\,{\rm cm\,s^{-1}}$ is the group velocity of Tuesday's waves, computed from1.2. ¿From1.2, the group velocity difference between the Sunday's and Tuesday's waves is

$$\Delta v_{\rm g} = \frac{g}{4\pi}\Delta T, \tag{1.5}$$

where $\Delta T = T_{\rm Sunday} - T_{\rm Tuesday}$ is the period difference. With $\Delta T \simeq 2.5$ s, we find

$$\Delta v_{\rm g} \simeq \frac{1000\,{\rm cm\,s^{-2}}}{4\times 3} \times 2.5\,{\rm s} \sim 200\,{\rm cm\,s^{-1}}. \tag{1.6}$$

So Tuesday's got ahead of Monday's by 200 cm every second. Using1.4 for the lag distance, and1.6 for the lag speed, we find the travel time is

$$\tau \sim L/\Delta v_{\rm g} \sim 1.1\cdot 10^6\,{\rm s} \sim \boxed{12\,{\rm days}}. \tag{1.7}$$

This time is how long ago the storm occurred.

The storm's distance from LA is simply

$$D \sim v_{\mathrm{g}}\tau = 1500\,\mathrm{cm\,s^{-1}} \times 1.1\cdot 10^6\,\mathrm{s} \sim 1.6\cdot 10^9\,\mathrm{cm} = \boxed{16\,000\,\mathrm{km}}. \tag{1.8}$$

That storm hit halfway around the world (the Pacific is big).

d) If the storm were a point, or an instant, then the waves would arrive perfectly sorted. For example, only 18 s waves would show up on Sunday, with no 17 s or 19 s waves mixed in. Let's first consider the maximum storm size, Δx (assuming the storm was very short for now). The 17 s and 19 s waves showed up together on Sunday, but 17 s waves move slower than 19 s waves. So the 19 s ones must have been generated farther from shore, so that they catch up to—but don't overtake—the 17 s ones in the 12 days of travel time. Using v_{g} computed from from1.5 (with $\Delta T = 2\,\mathrm{s}$) and τ from1.7,

$$\Delta x \sim (\Delta v_{\mathrm{g}})\tau \sim 150\,\mathrm{cm\,s^{-1}} \times 1.1\cdot 10^6\,\mathrm{s} \sim 1.6\cdot 10^8\,\mathrm{cm} = \boxed{1600\,\mathrm{km}}. \tag{1.9}$$

Now we assume that the storm happened at a point, but lasted for some time, Δt. The 17 s waves must have been generated before the 19 s ones, and the time between their starts must be large enough to allow the 19 s waves to catch up to the 17 s waves but not overtake them. We can write an equation for these words, and solve it for Δt, or more simply, we can estimate it from1.9: we convert Δx to a time, using the group velocity,

$$\Delta t \sim \Delta x / v_{\mathrm{g}} \sim \frac{1.6\cdot 10^8\,\mathrm{cm}}{1500\,\mathrm{cm\,s^{-1}}} \sim 10^6\,\mathrm{s} \sim \boxed{1\,\mathrm{day}}. \tag{1.10}$$

Or we could have started with say Sunday's fractional bandwidth, $f \equiv \Delta T/T \sim 0.1$, and used it to scale D and τ:

$$\Delta x \sim fD \sim 1600\,\mathrm{km}; \qquad \Delta t \sim f\tau \sim 1\,\mathrm{day}. \tag{1.11}$$

This method gives the same results as in1.9 and1.10 but is harder to follow.

Problem 2. *Seismic noise is dominated by forcing from ocean waves. Its power spectrum peaks near a period of 7 s, half the period of typical ocean waves. This frequency doubling is a nonlinear effect.*

a) *Identify its origin by comparing the time dependence of the horizontally averaged height of a deep body of water perturbed by surface travelling waves to that perturbed by surface standing waves. Note: don't confuse the average height of the surface with the average height of the water.*

b) *Estimate the amplitude of the bottom pressure variation in terms of the maximum height, ξ, of the waves.*

a) Consider the region of ocean above an area A of the ocean floor (here assumed to be horizontal). Let the depth of the ocean when no waves are upon it, be h. The mass of water in this volume is $m = \rho h A$. When no waves are present, the height of its center of mass is $\langle z\rangle = A\int_0^h \rho z dz/m = h/2$, and its gravitational potential energy is $W_0 = mg\langle z\rangle_0$. Now place upon the ocean surface a wave of amplitude $\eta(x,t)$. Let x be the wave propagation direction (or the direction along which it varies, for the standing wave), y be perpendicular to x along the surface, and z be the vertical direction. Now the height of the center of mass is

$$\langle z\rangle = \int_A \int_0^{h+\eta(x,t)(x,t)} \rho z\, dz/m \tag{1.12},$$

and the gravitational potential energy $W = mg\langle z\rangle$ is

$$
\begin{aligned}
W &= \int_A \left(\int_0^{h+\eta(x,t)} \rho g z\, dz\right) dx\, dy \\
&= \frac{1}{2}\rho g h A + \rho g h \int_A \eta(x,t)\, dx\, dy + \frac{1}{2}\rho g \int_A \eta^2\, dx\, dy \\
&\equiv W_0 + W_1 + W_2
\end{aligned}
\tag{1.13}
$$

(Since $dm = \rho\, dx\, dy\, dz$ is the mass of a volume element, and $\rho g z\, dm$ is its gravitational potential energy.) W_0 is the constant potential of the undisturbed ocean. W_1 is the part of the potential variation linear in the wave amplitude, and W_2 a nonlinear (quadratic) part.

As discussed in part (b) below, only long-range (of order the depth of the ocean) fluctuations in ocean height have effects at the bottom of the ocean (where we are interested in the pressure variations). So we integrate over a large surface. W_1 averages to zero. For a travelling wave, $\eta(x,t) = \xi\cos(kx - \omega t)$, so the remaining part of the averaged W is

$$
W_2 = \frac{1}{2}\rho g \xi^2 \int_S \cos^2(kx - \omega t)\, dx\, dy. \tag{1.14}
$$

Averaged over exactly one wavelength (or many wavelengths), $\cos^2(kx - \omega t) = 1/2$ (the ωt term is just a phase offset, but that doesn't affect the average of the cosine). So the difference in the average potential energy in our block of ocean with and without waves is

$$
\langle W\rangle - W_0 = \frac{1}{4}\rho g \xi^2 A, \tag{1.15}
$$

where A is the area of the surface of ocean we're considering (it'll drop out soon enough). The change in the height of its center of mass is $\langle z\rangle - \langle z\rangle_0 = (\langle W\rangle - W_0)/mg = \xi^2/(4h)$, which is independent of time. So the travelling wave doesn't cause any long-range time-varying fluctuations in the ocean height. But there is a non-zero average shift in the height of the water compared to water without a wave —the potential part of the wave energy— proportional to ξ^2.

The standing wave, where $\eta(x,t) = \xi\cos kx\cos\omega t$, is more interesting. Again W_1 averages to zero. But now putting this η into1.13,

$$
W_2 = \frac{1}{2}\rho g \xi^2 \int_S \cos^2 kx \cos^2 \omega t\, dx\, dy. \tag{1.16}
$$

Once again we take the average over many wavelengths so $\langle\cos^2 kx\rangle = 1/2$, and then

$$
\langle W\rangle = \frac{1}{4}\rho g \xi^2 A\cos^2\omega t = \frac{1}{8}\rho g \xi^2 A(1 + \cos 2\omega t) \quad \text{and} \quad \langle z\rangle - \langle z\rangle_0 = \frac{1}{8}\frac{\xi^2}{h}(1 + \cos 2\omega t)\,. \tag{1.17}
$$

The energy, and the height of the center of mass thus have a constant offset as for the ocean with a travelling wave, but now also have a time-dependent piece with frequency $\boxed{2\omega}$; this time-dependence accounts for the frequency doubled 7 s signals seen as seismic noise, since we now show that it can produce very long wavelength, coherent pressure pulses on the bottom of the ocean, which become waves in the rocky earth beneath ("seismic noise").

b) For shallow water waves [*not* the kind relevant in the ocean, where as we saw $P \sim 15\,\text{s}$ and $\lambda = P^2 g/(2\pi) \simeq 0.3\,\text{km}$, much smaller than the typical depth $h \sim 4\,\text{km}$ of the oceans], the horizontal motions of water caused by the waves are coherent across the entire depth. Hence the column of fluid underneath a small section of a wave must support the weight of the wave approximately hydrostatically, $\delta P \simeq \rho g \eta$. Thus in shallow water, the pressure variations at the bottom are linear in the wave amplitude.

In deep ocean ($h \gg \lambda/(2\pi)$), however, the linear part of the wave motion decays exponentially (with vertical scale length $\lambda/2\pi$) below the surface (one way to see this, mentioned in class, is to note that the linearised wave potential satisfies Laplace's equation, just like heat or potential variations whose effect also vanishes exponentially below a surface). This means that at the bottom, there is no term in the pressure variation linear in the wave amplitude as there is for shallow water waves. However, we saw in part (a) that for standing waves (but not travelling waves), the whole region containing the standing waves is moving up and down by an amount *second order* in the wave amplitude. Since the region can have a size $\gg h$, this motion does not decay exponentially below the surface. In the deep ocean, no horizontal flows are required, because although the center of mass of the water is moving up and down, this is only because water is being moved from the troughs to the crests of the waves at the surface: no net inflow or outflow of water on scales larger than a wavelength is needed.

From equation 1.17, the average center of mass of the whole large area A of the ocean containing a standing wave of amplitude ξ moves up and down by $\xi_{eff} = \xi^2/(8h)$ at frequency 2ω. The typical acceleration is thus $a = (2\omega)^2 \xi_{eff}$, and the external force required to move it $F = ma = \rho h A (2\omega)^2 \xi_{eff}$. The external force must be supplied by the pressure of the ocean floor over area A, so the fluctuating pressure at the ocean floor must satisfy $A\delta p = \rho h A (2\omega)^2 \xi^2/(8h)$, or

$$\boxed{\delta p \sim \tfrac{1}{2}\rho\omega^2\xi^2}. \tag{1.18}$$

Note that in deriving this, we have implicitly assumed that the water moves up and down as a rigid body -i.e. that it is incompressible, This assumption is marginal: the soundspeed in seawater is $1.5\,\text{km s}^{-1}$, so in a $3.5\,\text{s}$ up-down half of the $7\,\text{s}$ cycle a soundwave travels only $5\,\text{km} \sim h$. Numerically, with $\omega \sim 2\pi/14\text{s} \sim 0.5\text{rad s}^{-1}$ and $\xi \sim 100\,\text{cm}$, 1.18 gives

$$\delta p \sim 10^3\,\text{dyn cm}^{-2} \sim 10^{-3}\,\text{atm}.$$

The wavelength in rock ($c_s \sim 3\,\text{km s}^{-1}$) of a 7 s period wave is about 20 km, and the bulk modulus of rock $\mathcal{M} \sim 10^{11}\,\text{dyn cm}^{-2}$. Thus the fluctuating pressure of our pure standing wave would displace the ocean floor by about $(\lambda/2)\delta p/\mathcal{M} \sim 10^{-2}\,\text{cm}$.

Real data comparing ocean waves and the seismic waves they produce can be found in Haubrich, Munk & Snodgrass *Comparative Spectra of Microseisms and Swell*, in *Bull. Seismological Society of America*, **53**, 27-37 (1963). The power at 2ω (from the effect described above in the deep ocean) is at least 100 times that at ω (from waves breaking along the shallow beach). The actual seismic displacements after big storms are about $10^{-4}\,\text{cm}$, 10^2 times smaller than our pure standing wave would suggest. This is not too surprising: incoming waves are reflected at the shore with efficiency only ~ 0.2, and since at any given time ocean waves produced by a storm have a fractional bandwidth of about 0.1, only a small fraction of the waves are close enough in direction and wavenumber to produce a true standing wave with a horizontal scale comparable to the 20 km wavelength in rock [as we assumed in our estimate].

Problem 3. *A basketball is dropped onto a concrete pad from a high flying airplane.*

a) How high will it bounce?

b) Would your answer be any different if it were dropped from the top of Millikan library? A standard basketball has a radius of 12cm and a mass of 600g.

a) The basketball will reach terminal velocity long before hitting the ground. The drag force is balanced by gravity, so

$$\frac{1}{2} c_{\mathrm{d}} \rho v^2 A = mg, \tag{1.19}$$

where v is the terminal velocity, ρ is the air density, and A is the cross-sectional area of the ball. The height it would bounce back to, if all the energy were restored, is $h_{100} = v^2/2g$. Using1.19 to solve for $v^2/2g$, we find

$$h_{100} \sim \frac{m}{c_{\mathrm{d}} \rho A}. \tag{1.20}$$

The flow will be turbulent, but probably not enough for the boundary layer to become turbulent and stay attached (we'll check this at the end). So we'll use $c_{\mathrm{d}} \sim 1/2$. Putting in $m = 600\,\mathrm{g}$, and $A = \pi(12\,\mathrm{cm})^2$, we find

$$h_{100} \sim \frac{600\,\mathrm{g}}{0.5 \times 10^{-3}\,\mathrm{g\,cm^{-3}} \times \pi \times 144\,\mathrm{cm}^2} \sim 3 \cdot 10^3\,\mathrm{cm}. \tag{1.21}$$

Such a huge impact will dump lots of energy into heating the ground and basketball, and also some into the loud sound; maybe we get only 1/6 back. Then

$$h \sim \frac{h_{100}}{6} \sim 5\,\mathrm{m} = 1.5\,\mathrm{stories}. \tag{1.22}$$

Let's check the Reynolds number, as we promised. The terminal velocity is $v = \sqrt{2gh_{100}} \sim 2500\,\mathrm{cm\,s^{-1}}$. So

$$\mathrm{Re} \sim \frac{2Rv}{\nu} \sim \frac{24\,\mathrm{cm} \times 2500\,\mathrm{cm\,s^{-1}}}{0.2\,\mathrm{cm^2/\,s}} \sim 3 \cdot 10^5. \tag{1.23}$$

And actually for a rough sphere, the boundary layer goes turbulent—and the drag coefficient drops to 0.15—around $\mathrm{Re} \sim 1 \cdot 10^5$ (this reduction in drag is known as the *drag crisis*). Ato $\mathrm{Re} \sim 3 \cdot 10^5$, Sterl's sheet gives $c_{\mathrm{d}} \sim 0.2$. Therefore we must increase the h_{100} given in1.21 to

$$h_{100} \sim \frac{0.5}{0.2} \times 3 \cdot 10^3\,\mathrm{cm} \sim 80\,\mathrm{m}. \tag{1.24}$$

Maybe h is now $\sim h_{100}/8$, so we'll try

$$\boxed{h \sim 10\,\mathrm{m} = 3\,\mathrm{stories}}. \tag{1.25}$$

b) To reach terminal velocity, the ball must fall a distance roughly $h_{100} \sim 80\,\mathrm{m}$—somewhat higher than Millikan, which is 10 stories or 30 m. So we can, with only a moderate error, neglect air resistance in the fall from Millikan. If one-sixth of the energy comes back, we expect the ball to bounce to 10/6 stories, or $\boxed{1.5\,\mathrm{stories} = 5\,\mathrm{m}}$. Regulation basketballs must bounce 2/3 of the drop height, but that regulation is for 2 m drops.

Problem 4. *Of all the elements, only He, and to a lesser extent H_2 show interesting quantum effects in their liquid state. Give an order-of-magnitude calculation to explain why other liquids (including other noble gases besides He) don't show quantum effects, and are well described by classical (e.g. hard sphere) models. [Hint: some ingredients in your calculations will be binding energies, melting temperatures and the uncertainty principle]*

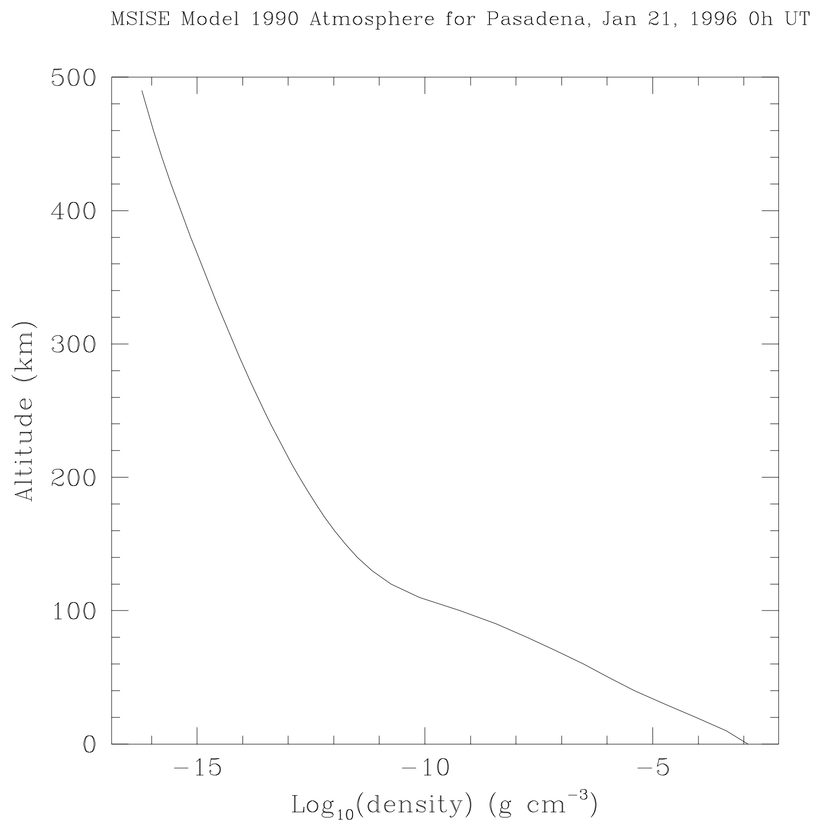
The momentum of an atom of mass m at temperature T is of order $p \sim \sqrt{3mkT}$. The de Broglie wavelength $\lambda \sim h/p$. In a liquid, the interatomic spacing is about the size of an atom a. When the de Broglie wavelength $\lambda \ll a$, atoms will collide as classical billiard balls. When $\lambda \gg a$, the atoms will interact coherently as quantum systems, and one might observe non-classical behaviour in the liquid. Notice that $\lambda \propto (mT)^{-1/2}$, so quantum behaviour is most likely to be observed for light atoms at low temperatures. The temperature can't be too low, however, or the material will solidify. So for a given material, the lowest temperature of interest is the melting temperature, T_m. As described in class, at low pressures $kT_m \sim U_b/10$, where U_b is the binding energy per atom in the solid. Therefore we would expect the lowest melting temperatures among the most weakly bound solids: i.e. those bound with only van der Waals forces—e.g. the noble gases, and molecules with no dipole moment. Looking up heats of sublimation in our favorite handbook, and noting that to within 20%, $1\,\text{kJ/mol} = 10^{-2}\,\text{eV/atom} = k(100\,\text{K})$, we find that this is indeed so:

Substance	$U_b/10k$(K)	T_m(K)	$\lambda = h/\sqrt{3mkT_m}$(Å)
H_2	10	14	5
He	1	—	13
Ne	20	24	1
Ar	75	84	0.4
Kr	100	116	0.26
Fe	4000	1810	0.08
C	8300	4123	0.12

Since atoms are all a few Å across (the inter-atomic spacing in both liquid He and liquid H_2 is 3.5Å), we see from the table that only for helium, and perhaps hydrogen are the de Broglie wavelengths in the liquid larger than the atoms. All other substances are too massive, and too tightly bound (hence solidify at too high a temperature) to be likely to show quantum effects. At sufficiently low temperatures $\ll T_m$, one can observe quantum effects in the sticking and hopping of gas vapor on solid surfaces for heavier materials, but these are much less spectacular than the quantum liquid effects. [NB: the quantum collective effects prevent He from solidifying at low pressures, though at high pressure it does solidify at $\sim 2\,\text{K}$, about as predicted from the $U_b/10$ rule. Of course spin and statistics affect the quantum states, so that the fermionic ^{3}He becomes a superfluid only at a temperature ($\sim$ a few millikelvin) much lower than the bosonic ^{4}He (which becomes a superfluid at 2.17 K). The phase diagrams for both isotopes are shown in the February 1987 issue of *Physics Today*, pp. 26 & 72.]

Problem 5. *Using the graph of the earth's atmospheric density as a function of altitude, estimate, for a meter-sized orbiting satellite,*

a) The altitude at which the mean-free path of air molecules is equal to the size of the satellite.
b) The altitude from which a satellite initially placed in a circular orbit would crash in 10 years.



c) *The altitude at which the satellite could be considered to have re-entered the atmosphere (time to crash becomes less than an orbital period).*

a) From Purcell's sheet, the mean free path of air molecules is $\lambda \approx 7 \times 10^{-6}(\rho_0/\rho)\,\mathrm{cm}$, where $\rho_0 \approx 10^{-3}\,\mathrm{g\,cm^{-3}}$ is the atmospheric density at sea level. (Note that the effective collision cross section, which can be calculated from this formula, is $\sigma \approx 7 \times 10^{-15}\,\mathrm{cm}^2$, significantly larger than the typical cross-sectional area of a molecule, $\sim (3\,\text{Å})^2 \sim 10^{-15}\,\mathrm{cm}^2$, because two molecules collide if their centers approach within two molecular radii.) For $\lambda = 100\,\mathrm{cm}$, we need $\rho \approx 7 \times 10^{-8}\rho_0 \approx 7 \times 10^{-11}\,\mathrm{g\,cm^{-3}}$. According to the graph, this density corresponds to an altitude of $\boxed{h \sim 110\,\mathrm{km}}$.

b) Let the satellite have radius R_{sat} and mass $M \sim (4\pi/3)R_{\mathrm{sat}}^3\rho_{\mathrm{s}}$, where ρ_{s} is the effective density of the satellite.

The satellite crashes because its energy is lost to air drag, and the rate of energy loss is

$$P \sim \frac{1}{2}\rho v^3 A, \tag{1.26}$$

where $A = \pi R_{\mathrm{sat}}^2$ is the cross-sectional area of the satellite. The mean free path will turn out to be much larger than the satellite (as we will soon see), and furthermore the satellite moves much faster than the thermal velocity of the molecules. So we're not dealing with fluid mechanics, where drag coefficients and Reynolds' numbers mean something. Instead the air molecule impacts are ballistic. The satellite transfers a bit of momentum to each molecule, and thereby loses energy, resulting in the drag formula1.26.

The atmosphere density, and therefore the energy loss rate, decreases exponentially with height. So most of the time in orbit will be spent descending the first scale height, H (the scale height is the height over which the atmosphere density changes by an e-fold). At a height h, the satellite has energy $E \sim Mgh/2$ (the 1/2 comes from the virial theorem). Therefore the energy loss in descending H is

$$\Delta E \sim MgH/2 \sim \frac{1}{2}\frac{4\pi}{3}\rho_{\mathrm{s}} R_{\mathrm{sat}}^3 gH. \tag{1.27}$$

The time, τ, to descend this height is the energy change divided by the energy loss rate. Using1.26 and1.27,

$$\tau \sim \frac{\Delta E}{P} \sim \frac{(2\pi/3)\rho_{\mathrm{s}} R_{\mathrm{sat}}^3 gH}{(1/2)\pi\rho v^3 R_{\mathrm{sat}}^2}. \tag{1.28}$$

Simplifying and solving for ρ (the air density),

$$\rho \sim \frac{(4/3)\rho_{\mathrm{s}} R_{\mathrm{sat}} gH}{\tau v^3}. \tag{1.29}$$

Since the orbital speed is $v = (gR_\oplus)^{1/2}$, we get

$$\rho \sim \frac{(4/3)\rho_{\mathrm{s}} R_{\mathrm{sat}} H}{\tau v R_\oplus}. \tag{1.30}$$

For high altitudes ($> 200\,\mathrm{km}$), the scale height is $H \sim 40\,\mathrm{km}$ (from taking the slope of the graph and converting to kilometers per e-fold).

Typically $R_{\mathrm{sat}} \sim 50\,\mathrm{cm}$, and $\rho_{\mathrm{s}} \sim 1\,\mathrm{g\,cm^{-3}}$. The orbital speed is roughly

$$v \sim \sqrt{1000\,\mathrm{cm\,s^{-2}} \times 6\cdot 10^8\,\mathrm{cm}} \sim 8\cdot 10^5\,\mathrm{cm\,s^{-1}}. \tag{1.31}$$

With $\tau \sim 3\cdot 10^8\,\mathrm{s}$ we get from1.30 that

$$\rho \sim \frac{1.3 \times 1\,\mathrm{g\,cm^{-3}} \times 50\,\mathrm{cm} \times 4\cdot 10^6\,\mathrm{cm}}{3\cdot 10^8\,\mathrm{s} \times 8\cdot 10^5\,\mathrm{cm\,s^{-1}} \times 6\cdot 10^8\,\mathrm{cm}} \sim 2\cdot 10^{-15}\,\mathrm{g\,cm^{-3}}.$$

(This is equivalent to requiring that the density is such that, during 10 years, the satellite will sweep a mass of air that is smaller by a factor $\sim H/R_\oplus$ than its own mass.) From the graph, the corresponding altitude is $\boxed{h \sim 400\,\mathrm{km}}$. At this density, the mean free path is roughly $3\cdot 10^4$ m (30 km!).

Notice that, although formally the Reynolds number for this flow is very small ($\nu \propto \lambda \propto \rho^{-1}$, approximately, so $\mathrm{Re} \propto \rho$), this statement is not very meaningful in a regime where the flow velocity, $v \sim 10^6\,\mathrm{cm\,s^{-1}}$, is much larger than the random velocity of the molecules, $v_{\mathrm{th}} < 3\cdot 10^4\,\mathrm{cm\,s^{-1}}$, and the mean free path, λ, is much larger than the relevant length scale, R_{sat}. In this regime the air does not really act like a fluid, but more like individual particles colliding with the satellite.

Note also that we (the TAs) misexplained this problem in section, stating that the satellite crashes when it uses up its $\sim mv^2$ kinetic energy. This statement is not correct: the satellite crashes with most of its kinetic energy, since it's only has to descend some small fraction ($H/R_\oplus$) of the earth's radius, and therefore only loses that same fraction of its energy (it's moving very fast when it hits the ground).

c) The orbital period is $\tau = 2\pi R_\oplus/v \sim 5 \cdot 10^3$ s, which is ~ 5 orders of magnitude less than the time considered in the previous part. Thus, the required atmospheric density is much higher than before, and we have to use a different scale height, $H \sim 6$ km (also from the graph). Plugging the new numbers into1.30, and using $\tau v = 2\pi R_\oplus$, we obtain

$$\rho \sim \frac{1.3 \times 1\,\mathrm{g\,cm^{-3}} \times 50\,\mathrm{cm} \times 6 \cdot 10^5\,\mathrm{cm}}{2 \times 3 \times (6 \cdot 10^8)^2\,\mathrm{cm}^2} \sim 1.6 \cdot 10^{-11}\,\mathrm{g\,cm^{-3}}. \tag{1.32}$$

From the graph, the corresponding altitude is $\boxed{h \sim 120\,\mathrm{km}}$ (which is similar to the altitude found in part a.)