Ph103b: Solutions to Problem Set 5

Problem 1. *Relate the speed at which a pole vaulter can sprint to the height he can vault. Provide both an analytical formula and a numerical estimate in your answer.*

Sprinting at speed v, the runner has kinetic energy $mv^2/2$. If the pole stores all the energy, and the vaulter converts it back into gravitational potential energy, then $mg\Delta h = mv^2/2$, so $\Delta h = v^2/2g$. Note that the symbol Δh was used for the height, not h, to emphasize that this height is the change in center-of-mass height of the vaulter, h_0. So $h = h_0 + v^2/2g$.

A world-class sprinter can run the 100 m dash in 9 s (this time includes accelerating at the start). So maybe for them $v \sim 12\,\mathrm{m\,s^{-1}}$, but perhaps carrying a pole, and being a notch off world-class sprinting performance, we have $v \sim 10\,\mathrm{m\,s^{-1}}$. We will take $h_0 \sim 1\,\mathrm{m}$. Then

$$h \sim 1\,\mathrm{m} + \frac{100\,\mathrm{m}^2}{2 \times 10\,\mathrm{m\,s^{-2}}} \sim 6\,\mathrm{m}. \tag{1.1}$$

But we have neglected many effects. First, the vaulter must reserve some kinetic energy so she or he moves forward over the bar. Second, some energy will be dissipated in the pole, and in the ground upon contact with the pole. Third, air drag will cost the vaulter a bit of energy on the way up. But the vaulter may use the Fosbury flop to get an extra few centimeters of clearance (by making the center of mass move under the bar a hair). And the vaulter can push off the pole with her or his arms on the way up. So maybe all the effects cancel out, and $h \sim 6\,\mathrm{m}$ is still reasonable.

The Ukrainian Sergey Bubka set a world record of 6.14 m (which stood as of a couple years ago).

Problem 2. *Skating: the coefficient of steel sliding on ice at temperatures between -11C and -5C is about* 0.005.

a) *Estimate the ratio of power a speed skater uses to overcome sliding friction compared to the power to overcome wind resistance (the world records in 5km and 10km speed skating are held by Koss, respectively* $6^{\mathrm{m}}35^{\mathrm{s}}$ *and* $13^{\mathrm{m}}30^{\mathrm{s}}$*).*

b) *Estimate the forward force that must be applied by the skating strokes to maintain speed against the total drag. You should find that this is large compared to the sliding friction, but small compared to the body weight. How is speed skating possible if the forward force must be large compared to the sliding friction (banana peel effect)? Are wind resistance and sliding friction the only relevant dissipation?*

a) The air drag is turbulent, so $F_{\mathrm{d}} \sim (1/2)\rho c_{\mathrm{d}} v^2 A$. Putting in $v \sim 1200\,\mathrm{cm\,s^{-1}}$; $A \sim 40\,\mathrm{cm} \times 100\,\mathrm{cm} = 4 \cdot 10^3\,\mathrm{cm}^2$; and $c_{\mathrm{d}} \sim 1$, we have

$$F_{\mathrm{d}} \sim \frac{1}{2} \times 1 \times 10^{-3}\,\mathrm{g\,cm^{-3}} \times 1.4 \cdot 10^6\,\mathrm{cm^2\,s^{-2}} \times 4 \cdot 10^3\,\mathrm{cm}^2 \sim 3 \cdot 10^6\,\mathrm{dyne}. \tag{1.2}$$

With $m \sim 80\,\mathrm{kg}$, the frictional force is

$$F_\mu \sim mg\mu \sim 8 \cdot 10^4\,\mathrm{g} \times 10^3\,\mathrm{cm\,s^{-2}} \times 0.005 \sim 4 \cdot 10^5\,\mathrm{dyne}. \tag{1.3}$$

So the power ratio, which is also the force ratio, is $F_\mu / F_{\mathrm{d}} \sim 1/7$.

b) The forward force, F, must balance the sum of the frictional and the air drag forces, so $F \sim 3.4 \cdot 10^6$ dyne. This force is large compared to the sliding friction (by a factor of 8) and small compared to the body weight (by a factor of $1/8\mu \sim 25$). Speed skating is possible because you cut into the ice with the edge of your blade, to give somewhere to push off. Otherwise you'd slip as if you were on a banana peel (try playing broomball in dress shoes, if you have strong bones).

Besides air resistance and sliding friction, there are a few other sources of dissipation. Your leg gives up its velocity (and therefore its kinetic energy) when your blade contacts the ice to dig in for the push. Some of this energy goes into tearing up the ice (the reason for ice grooming machines). Also you must do some work to move your legs back and forth (even if they didn't have to stop for a push), because muscles can't store energy perfectly.

Problem 3. *Estimate the electrical conductivity of sea water. The mass of salt per unit mass of seawater is 0.035. Hint: think of cages of water molecules and Stokes drag on spheres.*

Seawater is mostly water and table salt (NaCl). The ions feel a force from the electric field, $F = eE$ (ignoring signs). This force fights Stokes drag (the spheres are *very* small), so

$$6\pi\rho\nu R v = eE, \tag{1.4}$$

where R is the radius of the ion with its shell of water, and v is its terminal velocity. Since $J = nve = \sigma E$, where σ is the conductivity, we have

$$\sigma \sim \frac{ne^2}{6\pi\rho\nu R}. \tag{1.5}$$

We estimate $R \sim 2\,\text{Å}$ (a shell of water around each ion), and take $\nu \sim 10^{-2}\,\text{cm}^2\,\text{s}^{-1}$ as usual. A mole of NaCl, which provides two charges per molecule, is 60 g, so

$$n \sim 0.035\,\text{g}\,\text{cm}^{-3} \times \frac{1.2\cdot 10^{24}\ \text{charges}}{60\,\text{g}} \sim 7\cdot 10^{20}\ \text{charges}. \tag{1.6}$$

Then

$$\sigma \sim \frac{7\cdot 10^{20} \times 2.5\cdot 10^{-19}\,\text{esu}^2}{6\times 3\times 1\,\text{g}\,\text{cm}^{-3}\times 10^{-2}\,\text{cm}^2\,\text{s}^{-1}\times 2\cdot 10^{-8}\,\text{cm}} \sim 5\cdot 10^{10}\,\text{s}^{-1}. \tag{1.7}$$

Using the value from Purcell's sheet of $1\,\Omega^{-1} = 9\cdot 10^{11}\,\text{cm}\,\text{s}^{-1}$, the result is $\sigma \sim 0.05\,\Omega^{-1}\,\text{cm}^{-1}$, and the resistivity is $18\,\Omega\,\text{cm}$, fairly close to the value Purcell gives of $25\,\Omega\,\text{cm}$.

The biggest error here is in estimating the radius of the ion plus waters of hydration. Perhaps R should be greater than 2 Å, especially for sodium, which is smaller and therefore has a higher electric field at its 'surface'. A larger radius would reduce the conductivity. (In fact $R = 3\,\text{Å}$ would produce almost exactly the experimental conductivity.) Also charge transfer may increase the conductivity. Probably the continuum fluid dynamics approximation—using viscosity and the Stokes drag—is not so bad because the mean free path in water is so short, quite a bit shorter than an ionic radii, especially if the ion has a shell of water around it.

Problem 4. *Interplanetary Communication at Radio Frequencies*

a) *Calculate the power p received by an earth based radio telescope of diameter D from a spacecraft at distance s that transmits power P at wavelength λ using an onboard antenna of diameter d. Provide an analytical formula and a numerical evaluation in watts for $D = 70\,\text{m}$, $d = 3\,\text{m}$, $s = 10\,\text{AU}$, $\lambda = 4\,\text{cm}$, and $P = 10$ watts.*

b) *Denoting the system temperature of the earth based radio telescope by T, what is the maximum bandwidth $\Delta\nu$ at which the signal exceeds the noise? As for a), provide both an analytical formula and a numerical evaluation in Hz for $T = 20\,\mathrm{K}$.*

c) *Relate $\Delta\nu$ to the bit rate at which information can be transmitted from the spacecraft to earth.*

a) The angular spread in the transmitted beam is $\Delta\theta \sim \lambda/d$. By the time the beam reaches earth, it will have spread to a cone of size $L \sim s\Delta\theta \sim s\lambda/d$. The fraction of the beam power that the receiver captures is $f \sim (D/L)^2$, so

$$p = Pf \sim P\left(\frac{Dd}{s\lambda}\right)^2. \tag{1.8}$$

Putting in the numbers,

$$p \sim 10\,\mathrm{W} \times \left(\frac{70\,\mathrm{m} \times 3\,\mathrm{m}}{1.5 \cdot 10^{12}\,\mathrm{m} \times 0.04\,\mathrm{m}}\right)^2 \sim 10^{-16}\,\mathrm{W}. \tag{1.9}$$

b) The detector is some system with a degree of freedom, like a spring, so it gets a 'thermal' energy $\sim kT$ (thermal in quotes because the 'system temperature' is a faked-up temperature designed to give the right answer if we treat it as a thermal energy). If the spring has bandwidth $\Delta\nu$, then its damping time is $\tau \sim 1/\Delta\nu$, so the noise power is $kT/\tau \sim kT\Delta\nu$. We equate1.8 to the noise power, to find

$$\Delta\nu \sim p/kT \sim \frac{P}{kT}\left(\frac{Dd}{s\lambda}\right)^2. \tag{1.10}$$

Numerically, $kT \sim 3 \cdot 10^{-22}\,\mathrm{J}$, so $\Delta\nu \sim 300\,\mathrm{kHz}$.

c) To order of magnitude, the bit rate is roughly the bandwidth, so $\sim 3 \cdot 10^5$ bits/sec.

The most important effect we've neglected is probably antenna efficiency (about 0.7 for the Deep Space Net, and maybe less for the spacecraft antenna). So perhaps the actual bit rate is about $10^5\,\mathrm{s}^{-1}$. Water, which is in the atmosphere, doesn't absorb too badly at 4 cm. The numbers in this problem describe reasonably the Voyager spacecraft broadcasting from Saturn, from where it returned 40 000 bits per sec.

Problem 5. *Interplanetary Communication at Optical Frequencies*

a) *Repeat as for problem 4 except in this case the transmitter consists of a laser which emits $P = 1$ watt at $\lambda = 0.5\,\mu\mathrm{m}$ and is located at the focus of a 10 cm diameter mirror. Assume that the signal is collected by the Hubble Space Telescope which has $D = 2.4\,\mathrm{m}$.*

b) *What is the rate at which photons are received and how is that related to the bit rate at which information can be transmitted from spacecraft to earth.*

a) The analytical formula is identical to that in problem 4. Putting in the numbers,

$$p \sim 1\,\mathrm{W} \times \left(\frac{2.4\,\mathrm{m} \times 0.1\,\mathrm{m}}{1.5 \cdot 10^{12}\,\mathrm{m} \times 5 \cdot 10^{-7}\,\mathrm{m}}\right)^2 \sim 10^{-13}\,\mathrm{W}. \tag{1.11}$$

b) The energy per photon is

$$E = 2\pi\hbar c/\lambda \sim \frac{6 \times 0.2\,\mathrm{eV}\mu}{0.5\mu} \sim 2.4\,\mathrm{eV} \sim 4 \cdot 10^{-19}\,\mathrm{J}. \tag{1.12}$$

So the photon arrival rate is $\sim 250\,\text{kHz}$. Maybe we use 3 or 4 photons per bit (to fight the $\sqrt{N}$ photon shot noise), so we'll say the bit rate is $\sim 70\,\text{kHz}$.

Our worst error is probably in our estimate of the channel capacity (in the guess of 3 or 4 photons per bit).