Short Problems

#1) *How would you expect world records for weight lifted to scale with body weight?*

The mass one can lift (assuming standard competition weightlifting techniques such as the clean-and-jerk, which emphasise the thighs) is given by

$$M_{\text{lift}} = \frac{\epsilon P_{\text{muscle}}}{g} A, \tag{1}$$

where A is the cross-sectional area of the weightlifter's thighs, ϵ is the efficiency with which the force exerted by the muscles can be converted into vertical lift, and P_{muscle} is the pressure exerted by muscle, as discussed in class.

Assuming a homologous sequence of weightlifters (!), *i.e.*, all weightlifters have the same overall build regardless of size, then $M_{\text{thighs}} \propto M_{\text{person}}$. The mass in the thighs will determine the length scale of the thighs i.e. $V \sim L^3 \propto M$ and thus $L \propto M^{1/3}$. So $A \sim L^2 \propto M^{2/3}$ and

$$\boxed{M_{\text{lift}} \propto M_{\text{person}}^{2/3}} \tag{2}$$

The Guiness book of world records gives a comprehensive list of weightlifting records. In the men's category, scaling from the 54 kg to 108 kg division gives us a factor of 2 in mass and then supposedly a factor of 1.59 in mass lifted. In the category called the "Snatch", the ratio of mass lifted is $197.5/125 = 1.58$ and in the category "Jerk" it is $235/157.5 = 1.49$. In the women's category, we have a factor of 1.8 between the 46 kg and 83 kg divisions, yielding an expected ratio in mass lifted of 1.48. "Snatch" gives a mass-lifted ratio of $107.5/72.5 = 1.48$ and "Jerk" gives a ratio $127.5/92.5 = 1.38$. So our scaling is extremely good for the "Snatch" category and a slight overestimate for the "Jerk" category.

#2) *The speed of sound in liquid water* H_2O *at room temperature is* $1,482\,\mathrm{m/s}$. *Estimate the speed of sound in heavy water* D_2O *at the same temperature; be sure to state your assumptions and estimate the accuracy to which they ought to hold.*

The speed of sound in any ideal fluid is

$$c_s = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_S}. \tag{3}$$

Liquids, like solids, are nearly incompressible; the restoring force to a compression coming from distortions in atomic bonds (recall that water is slightly denser than ice!). The restoring force does *not* come from thermal energy as it does in a gas (so in liquids, c_s is *not* approximately $\sqrt{kT/m}$!). The speed of sound in a liquid is thus best estimated as

$$c_s \sim \left(\frac{H}{m}\right)^{1/2}, \tag{4}$$

where H is the heat of vaporization (binding energy per molecule in the liquid phase), and m is the molecular weight. To lowest order, changing the nuclear mass has no effect on the electronic states of the hydrogen bond, so H_2O and D_2O should have approximately the same H, but $m(D_2O) = (20/18)m(H_2O)$, so we would expect

$$c_s(D_2O) \sim \sqrt{\frac{18}{20}}c_s(H_2O) \sim 0.95c_s(H_2O) \sim \boxed{1406\,\mathrm{m/s}}. \tag{5}$$

In fact the speed of sound of D_2O at room temperature is 1384 m/s, 1.6% lower than the simple estimate. At boiling temperature, the heats of sublimation are $H(H_2O) = 40.66\,\mathrm{kJ/mol}$ and $H(D_2O) = 41.56\,\mathrm{kJ/mol}$, $\sqrt{H/m} = 1500\,\mathrm{m/s}$ for H_2O and $\sqrt{H/m} = 1440\,\mathrm{m/s}$ for D_2O. Notice that the magnitude of the difference in the H's (2%) is of the same order as the difference in the predicted sound speeds, but of the opposite sign. The higher heat of sublimation of D_2O can be understood as a consequence of the fact that that its hydrogen bonds have a lower zero-point energy (the ones to neighboring molecules, not the ones within the molecule, whose zero point energies are also much lower, but whose level spacings $\sim 0.1\,\mathrm{eV}$ are much larger than kT and the same in both liquid and gas phases, so the change doesn't affect the specific heat): $E_0 = (1/2)\hbar\omega_0$, where $\omega_0 = [H/(\mu a^2)]^{1/2}$, where $\mu = m_H m_O/(m_H + m_O) \simeq m_H$ ($m_H = 1, 2$ for H and D respectively), and $a = 2$Å is the length of the hydrogen bond. Thus $\omega_0 \sim (c_s/a)(m(H_2O)/\mu)^{1/2}$, and we find $E_0(\mathrm{H-O}) = 0.01\,\mathrm{eV}$ and $E_0(\mathrm{D-O}) = 0.007\,\mathrm{eV}$. With 2 bonds per molecule, we would expect the heat of vaporization of D_2O to be $2(E_0(\mathrm{H-O}) - E_0(\mathrm{D-O}) = 0.006\mathrm{eV} = 0.6\mathrm{kJ/mol}$ higher than that of H_2O due to the lowering of the zero-point energy, in reasonable agreement with the actual 0.9kJ/mol. Since these shifts are comparable to room temperature kT, the relation of these shifts to the sound speed is unfortunately not obvious. I have been unable find a good explanation of the sound speed effect in the literature —any takers?

#3) *Estimate how fast you could walk on the moon. [useful information: the moon's radius is 1/4 that of the earth, and its density of* $3\,\mathrm{gcm}^{-3}$ *is 0.6 that of earth.]*

$g = GM/R^2 \propto \rho R$, so that, relative to earth values,

$$\frac{g}{g_\oplus} = \frac{\rho}{\rho_\oplus}\left(\frac{R}{R_\oplus}\right) \sim 0.6 \times 0.25 \sim 0.15. \tag{6}$$

Thus $g \sim 150\,\mathrm{cm/s}^2$. When you walk, one foot is always on the ground, and during each stride your center of mass describes a circular arc with center located at the foot which is on the ground, and radius about equal to your leg length l. If the speed along the circular arc (which is the speed v at which you are walking) is such that $v^2/l > g$, the upward centrifugal acceleration would be larger than g —your foot would leave the ground, i.e., you would be running, not walking [notice that this condition corresponds to a Froude number, $\mathrm{Fr} \equiv v^2/gl > 1$; experiments indicate that in almost all animals the actual walk/run transition occurs slightly before the foot becomes weightless, at Fr=0.8]. So $v_{\max} \propto \sqrt{gl}$, and

$$\frac{v_{\max}}{v_{\max}(\oplus)} = \sqrt{\frac{g}{g_\oplus}} = 0.4. \tag{7}$$

So if you walk at 5 mph on earth, you could walk at $\boxed{2\,\mathrm{mph}}$ on the moon.

#4) *What fraction of the world's river water flowing out to sea is human urine?*

A sedentary person in a temperate climate consumes $\sim 2.5\,\ell$ of water per day (about half of that as obvious liquid, the other half as water in food, and water created by the metabolism of sugars and fats [whose burning produces H_20 and CO_2]). Personal observation suggests that $f \sim 0.3$ of this is passed as urine (the rest is sweated, exhaled or excreted in feces; people doing heavy exercise can consume $4\,\ell$ per hour, but most of this is sweated out to cool their bodies; however their urine production is somewhat larger too, since they have to metabolise more during exercise, so have more metabolic wastes to dilute). With about $6 \cdot 10^9$ people on earth, that means $5f \times 10^{15}\,\mathrm{cm}^3$ of human urine released per year across the world.

The average rainfall over the earth is 1 m/year, so the total rainfall over the fraction $\phi \sim 0.25$ of the earth's surface covered by land is

$$4\pi \times (6.3 \cdot 10^8\,\mathrm{cm})^2 \times 10^2\mathrm{cm\ y}^{-1} \times \phi \sim 10^{20}\,\mathrm{cm}^3\mathrm{y}^{-1}. \tag{8}$$

About half of this rainfall is evaporated, and half runs off in rivers, giving a net river runoff of $\sim 5 \times 10^{19} \mathrm{cm}^3 \mathrm{y}^{-1}$.

Thus, the fraction of the river water that is human urine is $\sim 1 \cdot 10^{-4} f$. Taking $f \sim 0.3$, we get $\boxed{\text{30 parts per million}}$.

#5) *How much electrical power (in MegaWatts) could be produced by burning in a conventional power plant all the junk mail received by everybody in the United States?*

Each person receives about 0.1 kg of Junk mail each day, and with $3 \cdot 10^8$ people in the US, that means $3 \cdot 10^{10}$ g of rubbish each day.

From Purcell's sheet, combustion releases 10^4 cal/g or $4 \cdot 10^{11}$ erg/g. Assuming an efficiency of conversion into useful power of about $\epsilon \sim 0.1$, we get that the power output is

$$P \sim 0.1 \times 4 \cdot 10^{11} \times 3 \cdot 10^{10} \sim 10^{21} \,\mathrm{erg/day} \sim 10^{16} \,\mathrm{erg/s} \sim \boxed{1\,\mathrm{GW}}. \tag{9}$$

#6) *A California Science Fair student proposes to mount magnets (all with N pole facing out) at the front and rear of cars to prevent collisions. Estimate the size of the smallest, cubical, permanent magnet necessary to prevent collisions at* 5 mph *and* 55 mph.

The kinetic energy of the car has to be converted into energy stored in the magnetic field. In cgs units, the energy density stored in a magnetic field B is $\sim B^2/8\pi$, so the total energy stored in volume V is $E \sim VB^2/8\pi$. Since $E \sim M_{\mathrm{car}} v^2/2$, we have

$$V \sim L^3 \sim \frac{4\pi M_{\mathrm{car}} v^2}{B^2}. \tag{10}$$

For a car of mass 10^6 g and speed 5 mph, or 220 cm/s, we have

$$L_{5\,\mathrm{mph}} \sim \left(\frac{4\pi \times 10^6 \times 220^2}{B^2} \right)^{1/3} \sim 20 \left(\frac{B}{10^4\,\mathrm{G}} \right)^{-2/3} \mathrm{cm}. \tag{11}$$

The best samarium-cobalt and neodymium-iron-boron permanent magnets have $B \sim 4000$ G (corresponding to alignment of 10% of the electron Bohr magnetons), which makes $\boxed{L \sim 40\,\mathrm{cm}}$ at 5 mph. The size scales as $v^{2/3}$, so for 55 mph, we have $\boxed{L \sim 200\,\mathrm{cm}}$. With the density of iron ~ 8 g/cc, these magnets would weigh 500 and 64,000 kg respectively. With a front and rear pair of magnets, these would increase the mass of the car by factors of 2 and 130 respectively, so the scheme does not seem very practical. In fact our calculation is not self-consistent, since the mass of the car [tank?] is now dominated by the magnet, so the kinetic energy has gone up requiring a bigger magnet. This iteration does not converge above a characteristic maximum speed which can be found by assuming that the magnet mass dominates that of the car: the energy stored in a magnet of size L is $E_M \sim L^3 B^2/8\pi$, and the maximum speed the magnet can have before its kinetic energy exceeds E_M is

$$v_{\max} \sim \left(\frac{L^3 B^2}{8\pi M} \right)^{1/2} \sim \frac{B}{(8\pi\rho)^{1/2}}. \tag{12}$$

For the quoted ρ and B, $\rho \sim 8$ g/cc and $B \sim 4000$ G, we have $v_{\max} \sim 288$ cm/s, or about 6 mph. Thus the scheme will not work above parking speeds with permanent magnets.

Superconducting electromagnets could get another factor of 10 in B, just barely allowing convergence in magnet mass at 55 mph, but you wouldn't want to be in the car if the cryogen supply failed.... There is also a start-up problem: since cars are made of steel, which is magnetised by external fields, if your car were magnet-equipped and the cars around you were not, tailgaters would be sucked up your tailpipe, and during parallel parking your car would inexorably attach itself to the closest car.

Long Problems

#1) *Overheated Orchestra*

During an orchestra concert, heat generated by the players, stage lights and the audience causes the temperature in the auditorium to rise by 5°K. Assuming the players take no corrective action,

a) *Estimate the fractional change in frequency of notes played by the wind instruments. Do the frequencies of their notes go up or down as the temperature rises?*

b) *Estimate the fractional change in frequency of notes played by the string instruments. Do the frequencies of their notes go up or down as the temperature rises? [hint: the coefficient of thermal expansion of spruce wood (used for piano and violin face and back plates) along the grain (which is parallel to the strings) is about 1/7 that of steel (used for strings).]*

a) The resonant frequency of a wind instrument of length L is given by $\nu = n(c_s/4L)$ if it has cylindrical bore (so the mouthpiece is a pressure maximum; clarinet, flute, trumpet, etc.) and by $\nu = n(c_s/2L)$ if it has conical bore (so the mouthpiece must also be nearly a node to avoid divergence of the pressure; oboe, bassoon, etc), with $n = 1, 2, 3$ for the various overtones blown. The temperature rise has two effects. First, it increases the speed of sound, and second, it lengthens the wind instrument:

$$\frac{\Delta\nu}{\nu} = \frac{\Delta c_s}{c_s} - \frac{\Delta L}{L} = \frac{1}{2}\frac{\Delta T}{T} - \alpha\Delta T\,,$$

where we have used $c_s = \sqrt{\gamma kT/m}$.

The fractional change in sound speed is $\frac{1}{2}\Delta T/T \sim 0.008$, which raises the frequency by this fraction. This change dominates the second term, the small frequency decrease from thermal expansion: from Purcell's sheet, the thermal expansion coefficient for solids, say brass, is $\alpha \sim 2 \cdot 10^{-5}$/deg. Thus for a trumpet, the change in length due to thermal expansion lowers the frequency $\alpha\Delta T \sim 10^{-4}$, an order of magnitude less than the change in sound speed. In wind instruments the wood grains are parallel to the length of the instrument, so from part (b), the thermal expansion coefficient is 1/7 of that of metals, and the effect of the change in length on the frequency is two orders of magnitude less than that of the change in sound speed. Thus the wind instruments' frequencies $\boxed{\text{rise} \sim 0.8\%}$.

b) The fundamental frequency of a string of length L at tension t and mass per unit length μ is

$$\nu = \frac{1}{2L}\sqrt{\frac{t}{\mu}}\,. \tag{13}$$

The tension is given by

$$\frac{t}{\mu} = \frac{B}{\rho}\frac{L(T) - L_0(T)}{L_0(T)}\,, \tag{14}$$

where $L_0(T)$ is the length the relaxed string would have, B is the elastic modulus, and ρ the density. Neglecting the small changes in B and ρ, differentiating gives

$$\frac{\Delta(t/\mu)}{(t\mu)} = \frac{\Delta L/L - \Delta L_0/L_0}{(L - L_0)/L_0}\,. \tag{15}$$

Assuming the wood of the instrument body is thick enough that it is not significantly compressed by the string tension, the ends of the string will be forced to move with the wood: $\Delta L/L = \alpha(\text{wood})\Delta T$, while $\Delta L_0/L_0 = \alpha(\text{steel})\Delta T$.

Before the temperature rose, the strings were stretched to the yield point of steel, so $\epsilon = (L - L_0)/L_0 \sim 0.005$ is the strain. Thus we have

$$\frac{\Delta\nu}{\nu} = -\alpha(\text{wood})\Delta T + \frac{1}{2\epsilon}\big[\alpha(\text{wood})\Delta T - \alpha(\text{steel})\Delta T\big]\,. \tag{16}$$

The first term on the left (the effect of the change in string length on the frequency) is negligible compared to the later terms (the effect of the change in string tension), and since $\alpha(\text{wood}) = \alpha(\text{steel})/7$, we finally get $\frac{\Delta\nu}{\nu} = -(3/7)\alpha(\text{steel})\Delta T/\epsilon$. With $\alpha(\text{steel}) \sim 1.4\cdot 10^{-5}/\text{K}$ and the strain at yield $\epsilon \sim 0.005$, we find that the string instruments' frequencies $\boxed{\text{fall} \sim 0.6\%}$.

This is not a bad estimate: according to E. Lieber (1982), *On the Tuning Stablity of Pianos*, *Das Musikinstrument* **31**, 602, the treble strings of a piano fall in pitch by -1.65 cent/K. A cent is 1/100 of an equal-tempered semitone, i.e. a frequency ratio $2^{1/1200} = 1.00058$, so $\Delta T = 5\,\text{K}$ changes the frequencies of the treble piano strings by $\Delta\nu/\nu = 0.00058 \times (-1.65) \times 5 = -0.5\%$, quite close to our estimate. According to Lieber's measurements, the tenor strings of the piano (which are overwound with wire not under tension to increase μ and thus lower ν) drop in frequency by -0.39 cent/K, or -0.1% for a 5 K rise in temperature. The piano thus gets noticeably out of tune with itself under such a temperature change (changes in humidity, which can swell the wood by 5%, are even worse), which is why concert pianos are tuned immediately before the concert.

#2) *House Lights*

A 100 watt light bulb is powered by household current flowing in a pair of parallel copper wires, 1 mm in radius, separated by 2 mm, and 2 m in length.

- *a) How much power is dissipated in the wires?*
- *b) At what velocity do the current carrying electrons drift?*
- *c) What is the magnetic field generated by the current 1 meter away from the wires?*

a) The current flowing through the light bulb is roughly $100\,\text{W}/120\,\text{V} = 0.83\,\text{A}$. This assumes that the power dissipated in the wires is negligible compared with that in the bulb itself, which we will see is correct. The power dissipated is $P = I^2R$, so we need to know R. [Note that it is not correct to use $P = V^2/R$ because we don't know V in the wires—most of the 120 V is dropped in the bulb filament.] The resistivity for copper is $\rho = 2\cdot 10^{-6}\,\Omega\cdot\text{cm}$. To get the resistance you simply note that the resistance scales with the wire length, and that leads us to a dimension of $\Omega\cdot\text{cm}^2$; so we need to divide by an area, A, which is the cross-sectional area of the wire. We end up with

$$R = \frac{\rho L}{\pi r^2} \sim \frac{2\cdot 10^{-6} \times 200}{3 \times 0.1^2} \sim 10^{-2}\,\Omega. \tag{17}$$

This is for one wire. So the power is then $P = 2I^2R \sim 2 \times (0.83)^2 \times 10^{-2} \sim \boxed{10^{-2}\,\text{W}}$.

b) Roughly 1 A flows in the wire, which is 1 Coulomb per second, or 10^{19} electron per second. The mean density of electrons is about $2n_a$, where n_a is the number density of atoms. To estimate n_a, we remember that $\rho \sim A/15$, where A is the atomic mass, and ρ is in g/cc. Then $n_a = N_A/15 \sim 4\cdot 10^{22}\,\text{cm}^{-3}$, where N_A is Avogadro's number. Then $n \sim 2n_a \sim 2 \times 4\cdot 10^{22} = 10^{23}\,\text{cm}^{-3}$. Also $A = \pi \times 0.1^2 \sim 0.03\,\text{cm}^2$. Since $I = 10^{19}\,/\text{s} = nvA$, we have

$$v = \frac{I}{nA} \sim \frac{10^{19}}{10^{23} \times 0.03} \sim \boxed{3\cdot 10^{-3}\,\text{cm/s}} \tag{18}$$

c) To find the magnetic field from *one* wire (there are two), use Ampere's law: $\int \mathbf{B}\cdot d\mathbf{l} = \mu_0 I$. As the Amperean loop, take a circle of radius r centered on the wire. $\mathbf{B}$ is completely tangential (for a long enough wire), so $2\pi r B = \mu_0 I$, or

$$B = \frac{\mu_0 I}{2\pi r}, \tag{19}$$

where we have reverted to SI units because the current is in amps. Note that in our problem we have two currents flowing in opposite directions, separated by $d = 2\,\mathrm{mm}$; their fields almost cancel (dipole effect). So we have to reduce B by a factor of d/r:

$$B = \frac{\mu_0 I d}{2\pi r^2} \sim \frac{4\pi \times 10^{-7} \times 1 \times 0.002}{2\pi \times 1^2} = 4 \cdot 10^{-10}\,\mathrm{T} = 4 \cdot 10^{-6}\,\mathrm{G}. \tag{20}$$

#3) *Hot Rocks*

This problem outlines an oversimplified calculation of how the temperature varies on the surface of the moon. Starting at $t = 0$, a smooth surface paved with cold ($T \approx 0\,\mathrm{K}$) black rocks is subjected to a constant flux, F, of optical radiation. Assume that the surface absorbs all of the incident radiation and radiates like a black-body at its surface temperature, T_s.

a) What is the equilibrium value, T_{eq}, of T_s?

b) Describe the time dependence of T_s before equilibrium is reached. Hint: The heat equation reads

$$\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T.$$

You are not expected to solve this equation.

c) What is the characteristic time, t_{eq}, for the approach to equilibrium, and what is the depth, δ, of the thermal boundary layer at this time?

d) Evaluate T_{eq}, t_{eq}, and δ for $F \approx 10^6\,\mathrm{erg\,cm^{-2}\,s^{-1}}$, a flux appropriate to solar heating of the moon.

Heat leaves the surface by black-body radiation into space and by conduction into the moon. As time flies, the thermal boundary layer (the thickness of warm rock, δ) steadily increases, which decreases the flux across the boundary layer.

a) In equilibrium all the moon-rock is an isotherm—no heat diffuses into the moon. So all the incident flux must leave by black-body radiation. Balancing the fluxes, $F = \sigma T_{\mathrm{eq}}^4$, or

$$\boxed{T_{\mathrm{eq}} = \left(\frac{F}{\sigma}\right)^{1/4}} \tag{21}$$

b) The approach to equilibrium is regulated by the flux through the thermal boundary layer, F_δ. As the boundary layer grows and F_δ approaches zero, $T_s \to T_{\mathrm{eq}}$. The inward flux is

$$F_\delta \sim K \frac{T_s}{\delta}, \tag{22}$$

where K is the thermal conductivity of rock. At time t, the boundary layer has thickness

$$\delta \sim \sqrt{\kappa t}, \tag{23}$$

where κ is the thermal diffusivity of moon rock. So

$$F_\delta \sim K \frac{T_s}{\sqrt{\kappa t}}, \tag{24}$$

The outward flux is σT_s^4, which varies very sharply with T_s. So initially, when T_s is small, most of the flux goes into the moon, and we may take $F_\delta \approx F$ in (24). Then

$$\boxed{T_s \sim F \frac{\sqrt{\kappa t}}{K}} \tag{25}$$

Stripping away the junk constants, we have $T_s \propto t^{1/2}$ initially. Much later, the surface temperature asymptotes to T_{eq}.

A more elegant method is due to Bradford Behr. After time t, the total energy dumped into an area A will be FAt. The energy is dumped into a volume $V \sim A\delta \sim A\sqrt{\kappa t}$. So the temperature rise will be $E/(V\rho c_p)$, which gives (25).

c) As mentioned above, two methods compete to remove the solar flux: diffusion into the rock, and radiation into space. Initially diffusion is the only possibility, and eventually, when the thermal boundary layer is sufficiently thick, diffusion is negligible and most of the heat leaves by radiation. The time, t_{eq}, when these two competing processes balance is the characteristic time for approach to equilibrium. Equating fluxes, $F_{\text{rad}} \sim F_\delta$ gives

$$\sigma T_s^4 \sim K\frac{T_s}{\sqrt{\kappa t_{\text{eq}}}}. \tag{26}$$

The temperature rises quite sharply in the beginning, so without gross error we may take $T_s \sim T_{\text{eq}}$ above. Using $\sigma T_{\text{eq}}^4 = F$, we have $F \sim KT_{\text{eq}}/\sqrt{\kappa t_{\text{eq}}}$, or

$$t_{\text{eq}} \sim \frac{K^2 T_{\text{eq}}^2}{F^2\kappa}. \tag{27}$$

Using the expression for T_{eq} in (21),

$$\boxed{t_{\text{eq}} \sim \frac{K^2}{F^{3/2}\sigma^{1/2}\kappa}} \tag{28}$$

At this time the thermal boundary layer will have depth

$$\boxed{\delta \sim \sqrt{\kappa t_{\text{eq}}} \sim KF^{-3/4}\sigma^{-1/4}} \tag{29}$$

d) From Purcell's sheet, in the "black body radiates…" section, $\sigma = 6\cdot 10^{-12}\,\text{W}\,\text{K}^{-4}\,\text{cm}^{-2} = 6\cdot 10^{-5}\,\text{erg}\,\text{K}^{-4}\,\text{cm}^{-2}$. For $F \sim 10^6\,\text{erg}\,\text{cm}^{-2}\,\text{s}^{-1}$, we have

$$T_{\text{eq}} = \left(\frac{10^6}{6\cdot 10^{-5}}\right)^{1/4} \sim \left(1.7\cdot 10^{10}\right)^{1/4} \sim \boxed{350\,\text{K}} \tag{30}$$

Moon rock is an insulator, for which Purcell gives $K \sim 10^{-2}\,\text{cal}/(\text{sec}\cdot\text{cm}\cdot\text{K})$. But this is too high (rock is a good insulator); the magic materials sheet says $K \sim 2\cdot 10^{-3}\,\text{cal}/(\text{sec}\cdot\text{cm}\cdot\text{K})$. A calorie is $4\cdot 10^7$ erg, so $K \sim 10^5\,\text{erg}\,\text{s}^{-1}\,\text{cm}^{-1}\,\text{K}^{-1}$. To find κ we use $K = \rho c_p \kappa$. Purcell's sheet gives ρc_p (the specific heat per volume) as $0.5\,\text{cal}\,\text{cm}^{-3}\,\text{K}^{-1}$. So we may take $\kappa = K/(\rho c_p) \sim 4\cdot 10^{-3}\,\text{cm}^2\,\text{s}^{-1}$. Then from (28), the characteristic time is

$$t_{\text{eq}} \sim \frac{\left(10^5\right)^2}{\left(10^6\right)^{3/2} \times \left(6\cdot 10^{-5}\right)^{1/2} \times 4\cdot 10^{-3}} \sim 3\cdot 10^5\,\text{s} \sim 4\,\text{days}. \tag{31}$$

After this much time, the thermal boundary layer has thickness $\boxed{\left(4\cdot 10^{-3} \times 3\cdot 10^5\right)^{1/2} \sim 30\,\text{cm}.}$

#4) *Boiling Water And Whistling Tea Kettles*

a) *It takes about 5 minutes to boil a liter of water on a kitchen stove.*
   i) How much power is used to heat the water?
   ii) At what rate does the boiling water evaporate?

b) *Many tea kettles come with whistles. The basic whistle is a hole of radius* $\approx 0.15\,\mathrm{cm}$ *through which water vapor can exit the kettle.*
   i) At what velocity does water vapor exit the hole when water is boiling inside the kettle?
   ii) What is the Reynolds number of the flow near the hole?
   iii) Why does the kettle whistle and what determines its frequency?
   iv) Estimate the radiated acoustic power.

a) *i)* Say the water starts out at 20°C. Raising the liter to to 100°C takes

$$E_{\mathrm{tot}} \sim 1000\,\mathrm{g} \times 1\,\mathrm{cal\,g^{-1}{}^\circ C^{-1}} \times 80^\circ\mathrm{C} \sim 8\cdot 10^4\,\mathrm{cal} \sim 3\cdot 10^{12}\,\mathrm{erg}. \tag{32}$$

If this is dumped into the water in 5 minutes, the power is $\boxed{P = E_{\mathrm{tot}}/300\,\mathrm{s} \sim 10^{10}\,\mathrm{erg/s} = 1\,\mathrm{kW}.}$

ii) The water evaporates at a rate $R = P/L_{\mathrm{vap}}$. From Purcell's sheet, $L_{\mathrm{vap}} \sim 10^4\,\mathrm{cal/mol} \sim 500\,\mathrm{cal/g}$, which is $L_{\mathrm{vap}} \sim 2\cdot 10^{10}\,\mathrm{erg/g}$. So $\boxed{R \sim 10^{10}/2\cdot 10^{10} = 0.5\,\mathrm{g/s}.}$

b) *i)* At STP, one mole of ideal gas vapor is 22.4ℓ. At 100°C a mole has more volume, by a factor of 1.3, so we'll take $30\,\ell/\mathrm{mol}$ as the conversion. At $18\,\mathrm{g/mol}$, our $0.5\,\mathrm{g}$ of water is $0.025\,\mathrm{mol}$, or $0.7\ell = 700\,\mathrm{cm}^3$. The flux, F, is therefore $700\,\mathrm{cm^3\,s^{-1}}$, and this is vA, where $A = \pi r^2$ is the cross-sectional area of the whistle hole, and v is the exit velocity. So

$$v \sim \frac{F}{A} = \frac{700}{\pi \times 0.15^2} \sim \boxed{10^4\,\mathrm{cm/s}} \tag{33}$$

ii) The Reynolds number is $\mathrm{Re} \sim rv/\nu$, where ν is the viscosity of steam, which we take as approximately that of air, $\nu \sim 0.2$. Then $\boxed{\mathrm{Re} \sim 0.15 \times 10^4/0.2 \sim 10^4}$. The flow is turbulent.

iii) The turbulent flow at the hole oscillates back and forth with velocity v, shedding vortices from side to side, in the famous von Karman vortex pattern. The angular frequency of oscillation is roughly the time to cross the whistle hole, so $\omega \sim v/2r$, and $f \sim v/4\pi r \sim 10^4/1.5 \sim 7\,\mathrm{kHz}$.

iv) The whistle is an acoustic monopole (there is mass flux). The efficiency of acoustic monopole radiation by turbulence is the Mach number, $M \equiv v/c_s$. The power density in the turbulent eddies is $\epsilon \sim \rho v^3/r$, so the acoustic power radiated is $P \sim M\rho v^3 r^2$. For the hot steam, $\rho = 1\,\mathrm{mol}/30\ell \sim 6\cdot 10^{-4}\,\mathrm{g/cc}$, and $c_s \sim 4\cdot 10^4\,\mathrm{cm/s}$. So

$$P \sim 0.25 \times 6\cdot 10^{-4} \times 10^{12} \times 0.15^2 \sim \boxed{0.4\,\mathrm{W}} \tag{34}$$

At a distance of 1 m, the intensity is $\sim 0.4/4\pi \sim 0.04\,\mathrm{W/m^2}$, which is about 105 dB ($1\,\mathrm{W/m^2}$ is 120 dB). The pain threshold is 120 dB, so 105 dB seems about right. At any rate, it's probably more accurate than the estimates used to derive it.

#5) *Resting and Bouncing Balls*

a) *A spherical rubber ball of density,* ρ, *shear modulus,* μ, *and radius,* R, *is lying on a smooth, rigid floor. Estimate the radius,* r, *of the circular area of contact between the ball and floor.*
   i) Use the Buckingham Π *theorem to show that* $r = RF(\rho g R/\mu)$.
   ii) Demonstrate that $F(x) \propto x^{1/3}$.

b) *The ball described in part a) is bounced on the floor with impact velocity* $v \gg (gR)^{1/2}$. *Estimate the contact time,* Δt, *between the ball and the floor.*

i) Use the Buckingham Π theorem to show that $\Delta t = (R/v)G(\rho v^2/\mu)$.
ii) Demonstrate that $G(x) \propto x^{2/5}$.

c) Numerical evaluation
i) Evaluate r in part a) for $R = 10\,\mathrm{cm}$, $\rho = 2\,\mathrm{g\,cm^{-3}}$, and $\mu = 10^7\,\mathrm{dyne\,cm^{-2}}$.
ii) Using the same parameters as for part a) together with $v = 5\,\mathrm{m/s}$, evaluate Δt and the maximum value of r in part b).

a) *i)* The relevant variables are ρ, μ, R, r and g. There are three dimensions associated with these variables, M, L and T. Thus, there are $5 - 3 = 2$ dimensionless Π variables. Two good choices are

$$\Pi_1 = \frac{r}{R} \qquad \text{and} \qquad \Pi_2 = \frac{\rho R g}{\mu}. \tag{35}$$

The Buckingham Pi theorem then tells us that

$$\boxed{\frac{r}{R} = F\left(\frac{\rho R g}{\mu}\right)} \tag{36}$$

ii) The downward force due to gravity is balanced by the shear force exerted by the compressed part of the ball. The pressure from the compressed part is $P \sim \mu\epsilon$, where ϵ is the strain, and the force is $F \sim Pr^2$. As we saw in the homework question about the train wheel (note the revised solution), $\epsilon \sim \delta/r$ where δ is the flattening and r is the radius of the contact surface. By the Pythagorean theorem, $R^2 = (R-\delta)^2 + r^2$ which implies $\delta \sim r^2/R$ for $\delta \ll R$. Hence $\epsilon \sim r/R$, $P \sim \mu r/R$ and $F \sim \mu r^3/R$.

The shear force balances gravity, $mg \sim \rho R^3 g$. So

$$\rho R^3 g \sim \frac{\mu r^3}{R}, \tag{37}$$

which implies

$$\boxed{r \sim R\left(\frac{\rho R g}{\mu}\right)^{1/3}} \tag{38}$$

b) *i)* In this case ρ, μ, R, v and Δt are the relevant variables, with the same three dimensions, so we again have 2 Π's. Two choices are

$$\Pi_1 = \frac{v\Delta t}{R} \qquad \text{and} \qquad \Pi_2 = \frac{\rho v^2}{\mu}. \tag{39}$$

Hence we get

$$\boxed{\Delta t = \frac{R}{v} G\left(\frac{\rho v^2}{\mu}\right)} \tag{40}$$

ii) From above, the shear force is $F \sim \mu r^3/R$. It acts through a distance δ, so the work done is $W \sim F\delta$, which is

$$W \sim F\delta \sim \frac{\mu r^3}{R} \times \frac{r^2}{R} \sim \frac{\mu r^5}{R^2}. \tag{41}$$

This work must be sufficient to stop the ball, so $W \sim mv^2 \sim \rho R^3 v^2$. Equating this to $\mu r^5/R^2$, we find

$$\left(\frac{r}{R}\right)^5 \sim \frac{\rho v^2}{\mu}. \tag{42}$$

The impulse from the shear force cancels out the initial momentum (it stops the ball). So $F\Delta t \sim mv$, and

$$\Delta t = \frac{mv}{F} \sim \frac{\rho R^3 \times v}{\mu r^3/R} \sim \frac{R}{v}\frac{\rho v^2}{\mu}\left(\frac{R}{r}\right)^3. \tag{43}$$

Using (42),

$$\boxed{\Delta t \sim \frac{R}{v}\left(\frac{\rho v^2}{\mu}\right)^{2/5}} \tag{44}$$

c) *i)* $R = 10\,\mathrm{cm}$, $\rho = 2\,\mathrm{g/cc}$, $\mu = 10^7\,\mathrm{dyn/cm}^2$, so from (38),

$$r \sim R \times \left(\frac{2 \times 10 \times 10^3}{10^7}\right)^{1/3} \sim 0.13R \sim \boxed{1.3\,\mathrm{cm}} \tag{45}$$

ii) $v = 5\,\mathrm{m/s}$ so from (44),

$$\Delta t \sim \frac{10}{500} \times \left(\frac{2 \times 500^2}{10^7}\right)^{2/5} \sim \boxed{6\,\mathrm{ms}} \tag{46}$$

and from (42), we have

$$\frac{r}{R} \sim \left(\frac{2 \times 500^2}{10^7}\right)^{1/5} \sim 0.55. \tag{47}$$

So $\boxed{r \sim 5.5\,\mathrm{cm}}$. The ball gets quite distorted.