Solutions to Problem Set 8

Problem 1. *Generation of Sound by Turbulence*

a) *Consider three-dimensional fluid turbulence with characteristic velocity v, outer scale L, and Mach number $M \sim v/c_s \ll 1$. What is the approximate amplitude of the turbulent pressure fluctuations?*

b) *Estimate the efficiency of acoustic radiation by the turbulence. Express the power radiated per unit volume as a fraction of the total energy dissipation rate per unit volume, namely $\epsilon \sim \rho v^3/L$. Hint: Quadrupoles are the lowest order acoustic multipoles.*

a) From Bernoulli, $\rho v^2 + p$ is a constant along a streamline. So velocity fluctuations $\sim v$ are caused by pressure fluctuations $\Delta p \sim \rho v^2$. In general, $c_s^2 = p/\rho$, so $\Delta p \sim p(v/c_s)^2 = pM^2$. For gases, the p is the gas pressure. For liquids, it is the bulk modulus. In problem set 6 we calculated the bulk modulus of water to be $B \sim 10^{10}\,\mathrm{erg/cm^3}$.

b) From the turbulence lecture, the velocity scales as $v_l/v_L \sim \beta^{1/3}$, where we've defined the dimensionless parameter $\beta \equiv l/L$. Sound is radiated because of the motion of the turbulent eddies. The frequency is set by how often the eddies cross the length scale, l, so $\omega_l \sim v_l/l$. From the 17 May lecture on sound (or the Pi theorem), the power radiated by a monopole of size l is

$$P_{\text{mono}}(l) \sim \rho\omega_l^2 v_l^2 l^4/c_s. \tag{1}$$

This is equation (35) from the typeset notes. Since $\omega_l \sim v_l/l$,

$$P_{\text{mono}} \sim \rho v_l^4 l^2/c_s. \tag{2}$$

In terms of the Mach number,

$$P_{\text{mono}} \sim \rho M_l^4 l^2 c_s^3, \tag{3}$$

where $M_l \equiv v_l/c_s$.

Mass injection (*e.g.*, a pulsing sphere) generates monopole radiation; momentum injection (*e.g.*, a ship wake) generates dipole radiation. Free turbulence has neither mass nor momentum injection (that's what the 'free' means), so the lowest allowed multipole is quadrupole. Actual examples of free turbulence are hard to come by; most turbulent flows have either mass or momentum injection.

But we'll assume that quadrupole radiation is the first allowed multipole. For a dipole, the pressure fluctuations are multiplied by the separation over the wavelength, $l/\lambda \sim v_l/c_s$, which is M_l. For a quadrupole, the factor is M_l^2. Since power is quadratic in the pressure fluctuations, the power in (3) needs to be scaled by M_l^4. So

$$P_{\text{quad}}(l) \sim \rho M_l^8 l^2 c_s^3. \tag{4}$$

From above, v and M_l scale as $\beta^{1/3}$. In a unit volume there are $N_l \propto l^{-3}$ sound sources, so $N_l/N_L \sim \beta^{-3}$. Putting in the scalings and accounting for the larger number of smaller eddies,

$$\frac{P_{\text{quad}}(l)}{P_{\text{quad}}(L)} \sim \frac{N_l}{N_L}\beta^{8/3}\beta^2 = \beta^{5/3} \propto k^{-5/3}, \tag{5}$$

where k is the wavenumber, $\sim l^{-1}$. The energy per mode also scales with $k^{-5/3}$, so this derivation probably contains in disguise the Kolmogorov argument given in class for the energy distribution. The upshot is that the long scales dominate the radiated sound (the noise).

From (4), the power radiated at the longest scale is

$$P \sim \rho M^8 L^2 c_s^3. \tag{6}$$

Per unit volume, this is $\mathcal{P} \sim \rho M^8 c_s^3/L$. The power density dissipated in the turbulence is $\epsilon \sim \rho v^3/L = \rho M^3 c_s^3/L$, so the efficiency of acoustic radiation is

$$\textit{Efficiency} \equiv \frac{\mathcal{P}}{\epsilon} \sim \frac{\rho M^8 c_s^3/L}{\rho M^3 c_s^3/L} = M^5. \tag{7}$$

Moral: fast turbulent flows generate a lot of sound. In most flows, where for example dipole radiation is important, the power law isn't so steep (M^3 instead), but speed still counts.

Problem 2. *Information Transfer Rates*

a) At what bit rate do you absorb information when reading a novel?
b) How long would it take to transmit over a video channel, bandwidth $\Delta\nu \approx 4 \times 10^6$ cycles per second, the information contained in the human genome, approximately one meter of DNA?

a) Let b be the number of bits per character of English, which we'll leave unknown for a moment. There are ~ 5 characters per word, and say you read 500 words per minute (about a page a minute). Then the bit rate, B, is

$$B \sim \frac{500\,\text{words per minute}}{60\,\text{sec/minute}} \times \frac{5\,\text{chars}}{\text{word}} \times \frac{b\,\text{bits}}{\text{char}} \sim 50b\,\text{Hz}.$$

Naively one might expect $b \sim \log_2 27 \sim 5$. But this ignores the rules of spelling, syntax and semantics, which greatly restrict the possible letter sequences. You don't often see `groo`, or more subtly, `I am growing to the store`. Taking all these restrictions into account is very difficult. Shannon described a beautifully simple method [1] to get upper and lower bounds on b. Some of the TA's and instructors have tried it out, [2] and find for novels (such as Mark Twain's *A Tramp Abroad*), $0.6 < b < 1.2$. So let say $b \sim 1$. Then $B \sim 50\,\text{Hz}$ (50 bits per second).

b) From the brain lecture (24 May), we know that base pairs are separated by 3 Å. [Each nucleotide is a ring, and the rings are stacked like pancakes in a (double) helix.] So one meter of DNA has $\sim 3 \cdot 10^9$ pairs per strand. But only one strand matters since the second is just the complement of the first—it contains no new information (as long as your repair and replication enzymes are working!) Each nucleotide could be one of four possibilities: adenine, guanine, cytosine or thymine. So if every sequence were equally likely, the number of bits per nucleotide would be $b = 2$. But actually it's a little less: of the $4^3 = 64$ amino acids that a triplet of bases could code for, only 20 are used. So $b \sim (log_2 20)/3 \sim 1.4$ is a better approximation. If we had someone who knew the grammar of DNA, we could use the method Shannon describes. But lacking such a person, we'll have to settle for $b \sim 1.5$. Then the total number of bits in the genome is $N \sim 1.5 \times 3 \cdot 10^9 \sim 5 \cdot 10^9$. Of course, to order of magnitude, it's irrelevant whether $b = 2$ or $b = 1$. The extra precision is included not because the answer requires it; but mainly to illustrate the importance of a grammar.

Through a channel of bandwidth $\Delta\nu$, one can transmit bits at a rate $\Delta\nu$. So it'll take $N/\Delta\nu \sim 5 \cdot 10^9/4 \cdot 10^6 \sim 1000\,\text{s}$ (15 minutes) to transmit the information.

Problem 3. *Information Content*

a) Estimate the number of distinct words that you can recognize. Explain your methodology.
b) How many distinct words would you estimate there are in the collected works of Shakespeare?

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[1] C. E. Shannon, "Prediction and entropy of printed English," *Bell System Tech. J.*, **30**:50–64 (1951).

[2] Anyone who wants to try it out, or is curious about the method, check the reference or send mail to `sanjoy@dope` for the program.

a) Out of 45 words chosen at random from the Webster's College Dictionary, this TA recognized 27 of them, for a hit rate of 60%. According to the dust jacket, the dictionary has "over 180,000 entries," so a reasonable estimate of the number of words recognized is $\sim 10^5$. The MIT linguist Steven Pinker quotes 60,000 as an average high school graduate's vocabulary. [3] Presumably this TA has filled his head with 40,000 otherwise useless words since high school (advection, diffusivity, *etc.*)

b) One uses far fewer words actively (speaking or writing) than one can recognize. In foreign language class, this TA was told that the active vocabulary is about 3 times larger than the passive. In Shakespeare's time, English wasn't so huge, so perhaps Shakespeare knew 50,000 words, and spoke or wrote with 10,000 or 20,000 words. With the advent Shakespeare's collected works on CD-ROM, actually counting the number of distinct words is fairly easy, and the result quoted by Pinker is ~15,000.

Problem 4.

Neglecting the small bell at the end, a clarinet with all the finger holes covered can be approximated by a cylinder $L = 65$cm long and $2a = 1.5$cm in interior diameter, driven by periodic pressure pulses at the mouth end. Consider frequencies close to the fundamental frequency (derived and illustrated in class). Your answers should not involve any quantities other than L, a, and physical properties of air (sound speed, thermal and viscous diffusivities).

a) *Estimate the Q (π times the number of wave cycles required to reduce the amplitude by $1/e$, or 2π times the number of wave cycles required reduce the power by $1/e$) of the clarinet. Be sure you consider heat transfer and viscous losses as well as radiative losses. Which dominate?*

b) *Neglecting the bell at the end, estimate the radiative efficiency [ratio of acoustic power out to power driving the pressure pulses at the mouth] of the cylindrical clarinet. How would this change if the diameter of the tube were doubled? What is the effect of the small flared bell at the end of a real clarinet?*

a) Sound waves are adiabatic—pressure changes produce temperature differences. Going radially outward from the long axis, the pressure change from to the sound wave approaches zero—viscosity eliminates the back-and-forth fluid motion at the cylinder wall. So the pressure, and therefore the temperature, vary with length scale a (in the radial direction).

In one radian of oscillation, the heat can diffuse a length $l \sim \sqrt{\kappa/(2\pi f)}$. From class, the fundamental frequency in a clarinet is $f \sim c_s/4L$, so $l \sim \sqrt{\kappa L/c_s}$. [From the kinetic theory, $\kappa \sim c_s l_m$, where l_m is the mean free path in air. So $l \sim \sqrt{l_m L}$. This geometric mean is a common occurrence—if you have a microscopic and a macroscopic length scale, a third useful one is often their geometric mean. For example, from the star twinkling problem in problem set 6, the Fresnel length is $\sqrt{\lambda D}$, where D is the atmosphere height and λ is the wavelength of light.] The temperature fluctuations have length scale a, so the viscous (or heat transfer) Q is

$$Q_{\text{heat}} \sim \frac{a}{l} \sim a \left(\frac{c_s}{\kappa L}\right)^{1/2}. \tag{8}$$

The heat also diffuses along the axis, but in that direction the pressure fluctuations vary on a length scale $\sim \lambda \sim L$. So the thermal losses from lateral diffusion will be much smaller than the losses from radial diffusion, by $a/L \ll 1$, and we will ignore them.

If the fluid elements move with characteristic velocity u, the energy stored in the clarinet cavity is

$$E \sim \rho u^2 a^2 L. \tag{9}$$

This formula is derived in the sound lecture (17 May). The air flux out the open end of the clarinet makes the sphere of air at the end pulsate in and out. So the open end of the clarinet

[3] Steven Pinker, *Language Instinct* (New York: W. Morrow and Co., 1994), pp. 149–150.

is a monopole (mass flux implies monopole, see the solution to problem 1). This sphere has radius a, and it oscillates with frequency ω and velocity u. From the second sound lecture, the energy radiated by the monopole is

$$P \sim 4\pi \frac{p}{c_s^3} u^2 \omega^2 R^4 \rho \frac{4\pi}{\gamma} u^2 \omega^2 a^4 / c_s. \tag{10}$$

This power is just equation (3) with the proper magic factor. The ratio E/P is the time constant for radiative decay of energy, and the Q for radiative decay is $Q_{\text{rad}} \sim E\omega/P$. So from (9) and (10), we find

$$Q_{\text{rad}} \sim \frac{E\omega}{P} \sim \frac{\gamma}{4\pi} \frac{\rho u^2 a^2 L\omega}{\rho u^2 \omega^2 a^4 / c_s} = \frac{\gamma}{4\pi} \frac{Lc_s}{a^2 \omega}. \tag{11}$$

Since $c_s/\omega \sim L$, we have

$$Q_{\text{rad}} \sim \frac{\gamma}{4\pi} \left(\frac{L}{a}\right)^2. \tag{12}$$

Putting numbers into (8) and (12),

$$Q_{\text{heat}} \sim 0.75 \times \left(\frac{3 \cdot 10^4}{0.2 \times 65}\right)^{1/2} \sim 40, \tag{13}$$

and

$$Q_{\text{rad}} \sim \frac{1.4}{12} \left(\frac{65}{0.75}\right)^2 \sim 10^3. \tag{14}$$

So $Q_{\text{rad}} \gg Q_{\text{heat}}$, which means heat losses dominate.

b) The total Q is the parallel combination of the two Q's above, which is dominated by Q_{heat}. [Q is a measure of resistance to energy escaping, so two escape paths combine their Q's like parallel resistors.] The radiative efficiency is $\sim Q/Q_{\text{rad}} \sim 0.04$. Since Q_{rad} scales as a^{-2}, and $Q \sim Q_{\text{heat}}$ scales as a, the radiative efficiency is $\propto a^3$. Doubling the diameter would increase the radiative efficiency by a factor of 8. But probably a clarinet this large would have finger holes too large to play on.

The flare at the end helps the impedance match to free space, thereby radiating more sound. This can be seen from (10). If we increase the area a^2 by a factor α, and preserve the same mass flux, then u decreases by α; so the power in (10) increases by α. To increase the radiated power, $\alpha > 1$—the bell is flared.

Problem 5. *The upper 3/4 of piano strings are bare steel wires, stretched to the yield point of steel.*

a) *Estimate the speed of transverse waves on such a piano string, and compare to the speed of sound in air.*

b) *Estimate (using only the properties of steel) the length of a piano string whose fundamental frequency ν_1 is middle C (262Hz).*

c) *The middle C piano string is about $d = 0.12$cm in diameter. The restoring force on a bent string has contributions from the differential stresses on its two sides ('stiffness' cf. the vibrating or buckling strut, present even if the center of the string is not in tension) and from the tension. Show that although the frequencies of the modes of a string of zero diameter are harmonically related (integer multiples n of the fundamental frequency ν_1), the stiffness term introduces an anharmonic term: $\nu_n = n\nu_1(1 + An^2)$, and estimate its coefficient A. By what percentage is the 4th harmonic of middle C sharp? Does the problem get better or worse for higher C notes? This* *effect means that beats will always be heard between the upper harmonics when octave intervals are played on a piano, no matter how it is tuned (to minimize the effect, piano tuners actually 'stretch' the tuning of octave strings, so their fundamental frequencies are not exactly a factor of 2 apart, but their overtones are more consonant).*

a) Transverse waves have velocity $c_t = \sqrt{T/\mu}$, where μ is the mass per unit length. This can be derived from the Pi theorem. The relevant variables are: the wave speed, c_t; the tension, T; the mass, M; and the length, L. Our string for now has zero thickness, so the only length scale is L. So we have four variables and three dimensions, therefore one Pi variable. A little fiddling gives $\Pi = c_t^2 M/TL$, so $c_t^{\sim} TL/M = T/\mu$, and $c_t \sim \sqrt{T/\mu}$. The magic constant here turns out to be unity.

Let A be the cross-section of string. The stress on the string is Y, the yield stress of steel, so the tension is $T \sim YA$. The mass per unit length is ρA, so $c_t = \sqrt{YA/\rho A} = \sqrt{Y/\rho}$. From the materials sheet, the yield stress for steel is $Y \sim 6 \cdot 10^9\,\mathrm{erg/cm}^3$, and $\rho \sim 8\,\mathrm{g/cc}$. So $c_s \sim 3 \cdot 10^4\,\mathrm{cm/s}$, which is about the speed of sound in air.

One may briefly wonder why c_t is not the same as the speed of sound in steel. The yield stress is defined as $Y = \epsilon B$, where B is the bulk modulus and ϵ is the yield strain. So $c_t = \epsilon^{1/2}\sqrt{B/\rho} = \epsilon^{1/2} c_s$, where c_s is the sound speed in steel. Since $\epsilon \sim 10^{-2}$, we find $c_t \sim c_s/10$. Compressional waves have a restoring force set by the interatomic forces, which are huge—so compressional waves move quickly. Transverse waves have a restoring force set by the tension. But you can't stretch most materials by anywhere near to the interatomic force—they flow or fracture long before that.

b) A piano string is fixed at both ends, so the lowest frequency (the fundamental) fits half a wavelength into the string length. Let λ be the wavelength and L be the string length. Then $\lambda = 2L$. Since $\lambda = c_t/f$, we have $L = c_t/2f \sim 3 \cdot 10^4/500 \sim 60\,\mathrm{cm}$.

c) Imagine one wavelength of the piano wire, of length $\lambda_n \equiv 2L/n$. When it is bent into one period of a sine wave the string will be tilted by some small angle θ, and the downward force will be $F \sim T\theta$. If the sine wave has amplitude y, we have $\theta \sim y/\lambda_n$, and $F \sim Ty/\lambda_n$. This restoring force acted over a distance $\sim y$, so the energy in the string due to tension is

$$E^{(0)} \sim Ty^2/\lambda_n. \tag{15}$$

But now we also have to worry about the energy due to the stiffness, since the string has been deflected by $\sim y$. The energy stored in a deflected beam was worked out the strength of materials lecture. From there (19 April, equation 17 of the typeset notes), the energy stored is

$$E^{(1)} \sim B\frac{d^4}{l_n^3}y^2. \tag{16}$$

This energy is a small perturbation on $E^{(0)}$. The energy ratio is

$$R \equiv \frac{E^{(0)}}{E^{(1)}} \sim \frac{Bd^4y^2/\lambda_n^3}{Ty^2/\lambda_n} \sim \frac{Bd^4}{T\lambda_n^2}. \tag{17}$$

Since $T = YA \sim \epsilon B d^2$, we have $R \sim (d/\lambda_n)^2/\epsilon$. Putting in $\lambda_n = 2L/n$, we find $R \sim (nd/2L)^2/\epsilon$. Energies scale as the square of the frequency, and the energy is perturbed by a small fraction, R. So the frequencies are perturbed by a small fraction, $\sim R/2$. This is an application of Taylor's theorem, or the binomial theorem,

$$\sqrt{1+x} \approx 1 + \frac{x}{2} + \cdots. \tag{18}$$

Ignoring stiffness, the n^{th} harmonic had frequency $n\nu_1$. So now it has frequency

$$\nu_n = n\nu_1 \left(1 + \frac{(d/L)^2}{8\epsilon} n^2\right). \tag{19}$$

In other words, $A \sim (d/L)^2/8\epsilon$. For our string, $\epsilon \sim 0.003$, $d = 0.12\,\text{cm}$ and $L = 60\,\text{cm}$. Putting this all in,

$$A \sim (0.12/60)^2/(8 \times 0.003) \sim 1.5 \cdot 10^{-4}.$$

So the fourth harmonic of middle C ($\nu_1 = 262\,\text{Hz}$) is sharp by a fraction $n^2 A = 16A \sim 0.2\%$. Higher C notes, made by shorter strings, will be more out-of-tune because $A \propto L^{-2} \propto \nu_1^2$.

One half-step is 6%, so this is one-thirtieth of a whole step. Two notes separated by such a small interval are probably indistinguishable when played a few minutes apart. But sounded together, they produce beats, at a frequency $\sim 0.002 f \sim 0.002 \times 4 \times 262 \sim 2\,\text{Hz}$. Beat notes can be heard into the fractions of Hertz, so this beat note is easy to hear.