Ph103b: Solutions to Problem Set 7

Problem 1. *Information Capacity Of A Continuous Channel*

a) *How many real numbers are required to characterize a continuous signal of length T and bandwidth W?*

b) *Estimate the number of significant bits each of these numbers carries if the signal is polluted by noise. Assume the signal to noise power ratio $S/N \gg 1$.*

c) *What value of S/N is required for operation of a 3×10^4 baud modem over a commercial phone line with $W = 3 \times 10^3\,\mathrm{Hz}$?*

a) A Fourier series for signal of duration T would have lowest frequency $f_0 \sim 1/T$. If the signal bandwidth is W, then we need $N \sim W/f_0 \sim WT$ Fourier amplitudes to characterize the signal. Whether these amplitudes are real or complex only introduces a factor of two (which we ignore). So we require $N \sim WT$ real numbers.

[An alternate method: The only dimensionless variable we can form from W and T is WT, so the number of real numbers required must be $N \sim (WT)^\alpha$, where α is some as yet unknown exponent. Since the amount of information transmitted will be proportional to the time T, we know $\alpha = 1$. Therefore $N \sim WT$ as before.]

b) If the signal and noise partition themselves similarly among the Fourier bands, then each band will have signal–to–noise ratio of S/N. We can distinguish S/N levels in each amplitude; to specify a particular level will take $\sim \log_2(S/N)$ bits, which is the number of significant bits carried by each Fourier amplitude.

c) So $W \log_2(S/N) \sim 3 \cdot 10^4$ bits/sec. With $W = 3\,\mathrm{kHz}$, we get $\log_2(S/N) \sim 10$ or $S/N \sim 1000$.

Problem 2. *More On Interplanetary Communication At Optical Frequencies Consider the communication system described in problem 5) of problem set 5. Would scattered sunlight be a significant noise source if the spacecraft were transmitting from in front of Saturn? In formulating your answer make sure to take into account that the space telescope resolves the disk of Saturn.*

The telescope picks up background light reflected off a an area of Saturn whose area is set by the diffraction limit of the Hubble. That area is $A \sim (\lambda s/D)^2$. The solar flux at Saturn is $F \sim L_\mathrm{s}/s^2$, where $L_\mathrm{s} \sim 4{\cdot}10^{33}$ erg/s is the solar power output (from Purcell's sheet). So the power reflected from Saturn, assuming perfect reflection, is $FA \sim L_\mathrm{s}(\lambda/D)^2$. Of that power only a fraction $f \sim (D/s)^2$ makes it into the telescope, so the power coming into the telescope is

$$P \sim fFA \sim L_\mathrm{s}\,(\lambda/s)^2 \sim 4 \cdot 10^{33}\,\mathrm{erg/s} \times \left(\frac{0.5 \cdot 10^{-4}\,\mathrm{cm}}{1.5 \cdot 10^{14}\,\mathrm{cm}}\right)^2 \sim 4 \cdot 10^{-4}\,\mathrm{erg/s}. \tag{1.1}$$

Saturn isn't perfectly reflecting, so we'll take $P \sim 10^{-4}\,\mathrm{erg/s} = 10^{-11}\,\mathrm{W}$.

Matters look bad: the laser power calculated in problem set 5 was only $10^{-13}\,\mathrm{W}$, which is about a factor of 100 fainter than the background light level we just estimated. Previously we assumed we'd use four laser photons to represent a bit (to beat the shot noise). And now we're getting 400 background photons on top of the four.

But the background light is broadband (it contains roughly an octave in frequency: the whole visible spectrum), while the laser has a narrow frequency range (even with the modulation required to send a signal on it). So we use a diffraction grating, or something fancier, to filter away most of the noise but preserve the signal. If the filter reduces the noise by a factor of 400, then there'll be only one background photon per four signal photons. The noise will then be low enough that we can still get one bit from the four signal photons.

[In fact, in the visible, the Hubble's spectrometer has a spectral resolution of $\lambda/\Delta\lambda \sim 2000$. The blackbody spectrum of the reflected sunlight has $\lambda/\Delta\lambda \sim 1$, so the spectrometer can cut the noise down by a factor of ~ 2000 if needed; our factor of 400 is therefore easily attainable.]

Problem 3. *Evaporation*

a) *Estimate the evaporation time per centimeter depth for water maintained at* 15 *degrees centigrade in vacuum. At this temperature the equilibrium vapor pressure is about* 13 mm *of Hg.*

b) *Estimate the evaporation time per centimeter depth as a function of wind speed for water maintained at* 15 *degrees centigrade in air. Consider a puddle of* 50 cm *diameter.*

c) *Compare your answers in a) and b) to the timescale over which puddles disappear in cloudy weather following a rain storm?*

a) The incoming mass flux would be (if the puddle were in equilibrium with its vapor) $\dot{M} \sim \rho v_\mathrm{T}/6$, where v_T is the thermal speed and the magic 6 accounts for five out of the six possible directions not resulting in a collision with the surface. [The actual factor, calculated by using the Maxwell velocity distribution to find the flux, is $\sqrt{6\pi} \approx 4.34$.] Assuming each collision results in capture, the outgoing flux must equal the incoming flux. In vacuum there's no incoming flux, but we can use still use $\dot{M}$ for the outgoing flux.

From the ideal gas law, we have $\rho = mP/kT$. We will find ρ by scaling it relative to ρ for air at STP. The mass provides a factor of $\sim 18\,\mathrm{amu}/30\,\mathrm{amu} \sim 0.6$, and the pressure provides a factor $13\,\mathrm{mm}/760\,\mathrm{mm} \sim 1/60$. (The temperature does almost nothing.) So $\rho \sim \rho_\mathrm{air} \times 0.6/60 \sim 10^{-5}\,\mathrm{g\,cm^{-3}}$. The thermal velocity is

$$v_\mathrm{T} \sim \left(\frac{3kT}{m}\right)^{1/2} \sim c\left(\frac{3kT}{mc^2}\right)^{1/2} \sim c\left(\frac{3\times 25\cdot 10^{-3}\,\mathrm{eV}}{18\cdot 10^{9}\,\mathrm{eV}}\right)^{1/2} \sim 2\cdot 10^{-6}c \sim 6\cdot 10^{4}\,\mathrm{cm\,s^{-1}}. \tag{1.2}$$

The liquid water volume flux is

$$\frac{\dot{M}}{\rho_\mathrm{water}} \sim \frac{\rho}{\rho_\mathrm{water}}\frac{v_\mathrm{T}}{6} \sim 10^{-5}\,\frac{v_\mathrm{T}}{6} \sim 0.1\,\mathrm{cm\,s^{-1}}. \tag{1.3}$$

Thus the puddle evaporates at $\sim 10\,\mathrm{s/\,cm}$.

b) In air, the water vapor must diffuse across a boundary layer, which has thickness $\delta \sim \sqrt{l\nu/v}$ where $l \sim 50\,\mathrm{cm}$ is the puddle size, and v is the wind speed near the ground. Assuming the air is dry outside the boundary layer (not entirely sound following rainy weather, but not too bad), the density difference across the boundary layer is $\Delta\rho \sim 10^{-5}\,\mathrm{g\,cm^{-3}}$. The mass flux across the layer is

$$\dot{M} \sim D\frac{\Delta\rho}{\delta}, \tag{1.4}$$

where D is the diffusion constant of water molecules in air. We will take $D \sim \nu \sim 0.2\,\mathrm{cm^2\,s^{-1}}$. Then the liquid water volume flux is

$$\dot{V} \sim \frac{\dot{M}}{\rho_\mathrm{water}} \sim \frac{\Delta\rho}{\rho_\mathrm{water}}\frac{\nu}{\delta} \sim \frac{\Delta\rho}{\rho_\mathrm{water}}\left(\frac{\nu v}{l}\right)^{1/2}. \tag{1.5}$$

Scaling $\dot{V}$ with respect to $l \sim 50\,\mathrm{cm}$ and $v \sim 100\,\mathrm{cm\,s^{-1}}$, we get

$$\dot{V} \sim 10^{-5} \left(\frac{v}{100\,\mathrm{cm\,s^{-1}}}\right)^{1/2} \left(\frac{l}{50\,\mathrm{cm}}\right)^{-1/2} \mathrm{cm\,s^{-1}}. \tag{1.6}$$

So the evaporation time per centimeter is (for $l = 50\,\mathrm{cm}$)

$$\tau \sim \left(\frac{v}{100\,\mathrm{cm\,s^{-1}}}\right)^{-1/2} \text{days}. \tag{1.7}$$

A reasonable wind speed is $v \sim 50$ or $100\,\mathrm{cm\,s^{-1}}$, so $\tau \sim 1$ day per centimeter.

c) In cloudy weather after a rainstorm, the thin puddles (maybe 0.5 cm deep) on a well-maintained tennis court disappear after a day or so. Thicker puddles on the road disappear after a couple days. These rates match those calculated in part (b), for evaporation limited by a boundary layer.

Problem 4. *Estimate the mean free path of a photon of blue light at sea level on a clear day. Scale your answer to obtain the mean free path of a similar photon propagating along a glass fiber.*

The scattering cross-section at angular frequency ω is

$$\sigma \sim \sigma_\mathrm{T} \left(\frac{\omega}{\omega_0}\right)^4, \tag{1.8}$$

where $\sigma_\mathrm{T} \sim 7 \cdot 10^{-25}\,\mathrm{cm}^2$ is the Thompson cross-section and ω_0 is the resonant frequency of an air molecule. For air the resonance is the in the near UV, maybe at $\hbar\omega_0 \sim 10\,\mathrm{eV}$, or $\lambda_0 \sim 2\pi\hbar c/10\,\mathrm{eV} \sim 120\,\mathrm{nm}$. Then at $\lambda \sim 480\,\mathrm{nm}$ (blue light),

$$\sigma \sim 7 \cdot 10^{-25}\,\mathrm{cm}^2 \times \left(\frac{120\,\mathrm{nm}}{480\,\mathrm{nm}}\right)^4 \sim 3 \cdot 10^{-27}\,\mathrm{cm}^2. \tag{1.9}$$

The mean free path is $l \sim 1/n\sigma$, where n is the number density of outer-level electrons. Assuming one electron per molecule of N_2 or O_2, we have $n \sim 6 \cdot 10^{23}/2 \cdot 10^4\,\mathrm{cm}^3 \sim 3 \cdot 10^{19}\,\mathrm{cm}^{-3}$. Then $l \sim 10^7\,\mathrm{cm} = 100\,\mathrm{km}$.

The number density of scattering electrons in glass is roughly $1/\left(3\,\text{Å}\right)^3 \sim 3 \cdot 10^{22}\,\mathrm{cm}^{-3}$, roughly a factor of 1000 larger than that in air. So $l \sim l_\mathrm{air}/1000 \sim 0.1\,\mathrm{km}$. But the short-range order will reduce l. If N electrons oscillate coherently, then they will act as a single electron of charge $q' = Ne$. The number density of these scattering centers is $n' = n/N$, and the power radiated goes as $P' \propto n'q'^2$; so $P' = NP$. Thus the scattering cross-section will go up by a factor of N and the mean free path will fall by a factor of N. Taking say $N \sim 4$, we get $l \sim 0.02\,\mathrm{km} = 20\,\mathrm{m}$.

[For amusement, we include some actual data to compare our estimates with. The extrapolated l for a low-loss fiber at 480 nm is $l \sim 0.16\,\mathrm{km}$ (extrapolated from the data included below). Perhaps such low-loss fibers have very little short-range order. Some data on a 'typical, low-loss, high-silica fiber', kindly provided by E. Pettit of Aerovironment, Inc., follows:

λ (nm)	*UV Loss (dB/km)*	*Rayleigh Loss (dB/km)*
1100	0.03	0.78
1000	0.125	1.19
900	0.19	1.81
800	0.31	3.06
700	0.73	5.19

The mean free path is the distance to lose $\approx 4.3\,\mathrm{dB}$ (an e-fold attenuation in power). The scattering resonance is at $\lambda = 140\,\mathrm{nm}$ (a little longer than our guess of $120\,\mathrm{nm}$). The UV loss is from the damping: the energy that the oxygen valence electrons absorb and lose nonradiatively (thereby heating the fiber).

At the useful wavelength of $1330\,\mathrm{nm}$, repeaters (booster stages) are actually placed every $\sim 25\,\mathrm{km}$. We can estimate what their spacing should be by scaling our Rayleigh scattering loss estimate at $480\,\mathrm{nm}$. Away from the resonance, we have $l \propto \lambda^4$ so

$$l_{1330} \sim l_{480}\left(\frac{1330}{480}\right)^4 \sim 0.1\,\mathrm{km} \times 60 \sim 6\,\mathrm{km}. \tag{1.10}$$

So $6\,\mathrm{km}$ is the e-fold attenuation length; if we assume the repeater is quite sensitive, and can tolerate say a power loss factor of $e^{10} \sim 30\,000$, then we'd have to put repeaters every $60\,\mathrm{km}$. Allowing for a reasonable safety margin, short-range order, UV, and other loss mechanisms, the actual value of $25\,\mathrm{km}$ seems reasonable.]

Problem 5. *A ventilation system steadily blows fresh air at temperature 18 C (65 F) into a room. If there are enough people in the room to maintain the temperature at a steady state value of 24 C (75 F), by what fraction is the air leaving the room depleted in oxygen?*

If $\dot{M}$ is the air mass exchange rate, the power produced by the people is $P \sim \dot{M} c_{\mathrm{p}} \Delta T$. Burning carbohydrate is

$$\mathrm{CH_2O} + \mathrm{O_2} \quad \rightarrow \quad \mathrm{CO_2} + \mathrm{H_2O} + E, \tag{1.11}$$

where E is roughly $4\,\mathrm{kcal}$ per gram of $\mathrm{CH_2O}$, or roughly $4\,\mathrm{kcal/g}$ of $\mathrm{O_2}$ since $\mathrm{CH_2O}$ and $\mathrm{O_2}$ have almost the same molar mass. So the mass rate of oxygen removal is $\dot{M}_{\mathrm{O_2}} \sim P/E \sim \dot{M} c_{\mathrm{p}} \Delta T / E$. Since oxygen makes up ~ 0.2 of the atmosphere by mass, the depletion fraction is

$$f \sim \frac{\dot{M}_{\mathrm{O_2}}}{0.2\dot{M}} \sim \frac{5 c_{\mathrm{p}} \Delta T}{E}. \tag{1.12}$$

With $c_{\mathrm{p}} = (7/2)R \sim 7\,\mathrm{cal\,mol^{-1}\,{}^\circ C^{-1}}$, and taking the molar mass as $30\,\mathrm{g}$, we have

$$f \sim \frac{5 \times 7 \times 6/30}{4000} \sim 2 \cdot 10^{-3} = 0.2\%. \tag{1.13}$$

Since the ratio of the $\mathrm{CO_2}$ to the $\mathrm{O_2}$ content of air is $0.2/0.00035 \sim 600$, the $\mathrm{CO_2}$ concentration will be increased by $600f \sim 1$: it will double. Probably the stuffiness comes from the extra $\mathrm{CO_2}$, not the lack of $\mathrm{O_2}$. [If the $\mathrm{CO_2}$ concentration in the whole atmosphere doubles, what'll be worse, the stuffy atmosphere, or the high sea level?]