Solutions to Problem Set 4

1. The temperature in the Earth's crust increases at a rate of $20^\circ K$ per kilometer of depth.
   a) How cold would the Earth become with the sun turned off?
   b) Could geothermal sources provide a solution to the world's energy problem?

   From the thermal conductivity equation, the energy flux is

$$J = k\frac{dT}{dr}.$$

Since rock is an insulator, the conductivity is about 10^{-2} cal/s-cm-°K, so this becomes

$$\begin{aligned} J &\approx 10^{-2}\frac{\text{cal}}{\text{s cm}^\circ\text{K}}\frac{20\text{K}}{10^5\text{cm}} \\ &\approx 80\frac{\text{erg}}{\text{cm}^2\text{s}} \end{aligned}.$$

If we assume that the Earth radiates as a blackbody, the energy flux is also given by the Stefan-Boltzmann law

$$J = \sigma T^4 = 5.67\times 10^{-5}T^4.$$

Setting the two expressions equal, this gives T∼30K.

The Power from geothermal heat is

$$4\pi R_\oplus^2 J \sim 4\times 10^{13}\text{Watts}.$$

Whereas from the Global 2000 Tecnical Report the amount of energy used in 1990 is 384×10^{15}Btu where a Btu is 1055 Joules.

$$\frac{384\times 10^{15}\text{Btu}}{\text{year}} \sim 1\times 10^{13}\text{Watts}$$

It looks pretty unlikely that like geothermal sources could solve the world's energy problem.

2. Consider the properties of iron at low temperature, but at the high pressure,

$$P \approx \frac{m_e^4 c^5}{\hbar^3} \approx 10^{25}\,\text{dyne}\,\text{cm}^{-2}.$$

Such conditions are met in the cores of massive white dwarfs and in the crusts of neutron stars. Estimate:
   a) the mass density,
   b) the elastic shear modulus at temperatures well below the melting temperature,

c) the melting temperature,

d) the debye temperature,

e) the electrical conductivity at the melting temperature.

a) At pressures like this, much above the zero pressure bulk modulus, and low temperature, the pressure is due mostly to electron degeneracy (Fermi) pressure, $P \sim n_e \epsilon_F$, where ϵ_F is the Fermi energy. Initially assume the electrons are non-relativistic, so $\epsilon_F \sim p_F^2/(2m_e) \sim (\hbar n_e^{1/3})^2/(2m_e)$. Equating to the specified P then gives

$$n_e \sim \left(\frac{m_e c}{\hbar}\right)^3 \sim \Lambda_c^{-3}\,, \quad \epsilon_F \sim m_e c^2\,.$$

Here Λ_c is the Compton wavelength of the electron. Thus the electrons are just barely relativistic (one gets the same answer if one assumes at the start that the electrons are relativistic, $\epsilon_F \sim p_F c \sim \hbar n_e^{1/3} c$). Thus $\rho = A m_p n_e/Z \sim 2 m_p n_e \sim 10^8\,\mathrm{g\ cm^{-3}}$.

b) $E \sim P$. The *shear* modulus μ is determined by the lattice strength though, and is much lower: $\mu \sim Z^2 e^2 (n_e/Z)^{2/3} \sim 10^{24}\,\mathrm{dyn\ cm^{-2}}$.

c) The lattice melts at temperature T_m (note that the electrons are already a gas, so they don't have to melt!) when the thermal energy of the ions is some small fraction $f \sim 0.01$ of the ion binding energy. Since the electrostatic shielding length $\sim (\epsilon_F/4\pi n_e e^2)^{1/2} \sim r_i Z^{1/3} \alpha^{-1/2}$ is larger than the lattice spacing $r_i \sim (n_e/Z)^{-1/3}$, the ions see each other's full charge. Thus $kT_m \sim f Z^2 e^2/r_i \sim \alpha Z^{5/3} m_e c^2$. Thus $T_m \sim 10^8 (f/0.01)\,\mathrm{K}$.

d) The Debye temperature T_D (at which all oscillations are thermally excited) is given by $kT_D \sim \hbar\omega_D$, where ω_D is the maximum frequency of lattice vibrations. The maximum frequency of ion acoustic waves is the ion plasma frequency $\omega_{pi}^2 \sim 4\pi n_e e^2 Z/m_{Fe}$. Thus $kT_D \sim \hbar\omega_{pi}$, and $T_D \sim 5 \times 10^6\,\mathrm{K}$. Note that this result can also be derived by writing $\omega_D \sim v k_{\mathrm{max}} \sim 2\pi v/r_i$, with $v \sim [(Z^2 e^2/r_i)/m_{Fe}]^{1/2}$, the velocity of waves whose restoring force is provided by the Coulomb lattice (not the electron pressure!).

e) If we ignored the Pauli exclusion principle and pretended that all the electrons in the metal were free to be in any quantum state, the conductivity would be $\sigma = j/E = n_e e v_D/E = n_e e (eE\tau/m_e)/E = n_e e^2 \tau/m_e$, where v_D is the mean electron drift velocity in the applied electric field, and $\tau = \lambda/v_F$ is the mean free time between phonon-electron scattering at the Fermi surface, with Fermi velocity $v_F \sim c$. But since $kT_m \ll \epsilon_F$, we shouldn't ignore the exclusion principle.

Alternatively, we could include the electric potential in the free electron hamiltonian. Then the whole Fermi sea of state would be accelerated uniformly at eE/m_e. The exclusion principle comes in because the scattering rate is appreciable only for electrons within kT of the Fermi surface. This scattering of electrons from the front to the back of the Fermi sphere results in an effective velocity of the whole Fermi sphere of $eE\tau(v_F)/m_e$, again giving the same conductivity as if we had ignored the Pauli principle.

At the melting temperature, the ions are almost completely disordered, so the electron-phonon mean free path $\lambda \sim r_i$ (for lower temperatures $T_D < T < T_m$, $\lambda \sim r_i(T_m/T)$).

Thus

$$\sigma \sim \frac{n_e e^2}{m_e} \frac{Z^{1/3} n_e^{-1/3}}{c} \frac{T_m}{T} \sim \alpha Z^{1/3} \frac{c}{\Lambda_c} \frac{T_m}{T} \sim 10^{19} \frac{T_m}{T} \,\mathrm{s}^{-1} \sim 10^7 \frac{T_m}{T} \mathrm{ohm\ cm}^{-1} \,.$$

Thus even at the lattice melting temperature, the conductivity is an order of magnitude higher than that of copper at room temperature $\sim 6 \times 10^{17}\,\mathrm{s}^{-1}$.

3. A basketball is dropped onto a concrete pad from a high flying airplane.
   a) How high will it bounce?
   b) Would your answer be different if it were dropped from the top of Millikan library?

   To find out how high it will bounce, we need to know what the terminal velocity of a basketball is. The drag force is

$$F_{\text{drag}} = c_D \frac{\rho_{\text{air}} v^2}{2} \pi r_{\text{ball}}^2$$

where c_D, the coefficient of drag, is ~ 1 at high Reynolds numbers.

A standard basketball has a radius of 12cm and a mass of 600g, so terminal velocity is reached when

$$F_{\text{grav}} \approx F_{\text{drag}}$$

$$mg = c_D \frac{\rho_{\text{air}} v^2}{2} \pi r_{\text{ball}}^2$$

$$\Rightarrow v = \left(\frac{2mg}{c_D \pi \rho_{\text{air}} r_{\text{ball}}^2} \right)^{1/2} \approx 14\text{m/s}.$$

The bounce height will be given roughly by $h = \frac{v^2}{2g} \approx 10\text{m}$ for a perfectly elastic bounce. The coefficient of restitution of a basketball (meaning the ratio of the energy after the bounce to the energy before the bounce) is about 0.5, and the drag force on the ball on the way up will slow it down, so a good order of magnitude estimate is $\leq 4\text{m}$ for the bounce. (Also the coefficient of restitution may be a bit lower for larger velocities.) This will be the height of a bounce from an airplane.

Since from Millikan the vacuum free-fall velocity is only $\approx$24m/s, (assuming a height of ≈ 30 m), the ball may not have had time to reach its true terminal velocity. It is therefore possible that the ball when dropped from Millikan will not get quite as high as it would if it were dropped from an airplane.

4. [modification of Eric Dickson's problem 6 from last week]

   By 1618, tensions between the ruling Roman Catholics and the Protestants in Bohemia had risen to a fevered pitch, culminating in the ejection of the two (Catholic) imperial regents from an upper-floor window of the Prague castle. This event, known as the Defenestration* of Prague, began the Thirty Years' War.

* Latin *de*: from + *fenestra*: window

The two Catholics survived the fall without serious injury, and attributed their good fortune to the favor of the Lord. Skeptical Protestants pointed out that the Catholics had landed in a large pile of horse manure which had cushioned their fall.

a) Assume for the moment that the pile of horse manure was infinitely deep. Use the fact that the Catholics were not injured to estimate an upper limit to the viscosity of the horse manure. [optional: if you have experience with horse manure, discuss the probable age and temperature of manure with this viscosity]

b) Now assume that the viscosity is far below your limit in (a). Estimate the depth of manure needed to provide saving grace for the fallen Catholics.

c) If the Catholics had instead fallen from heaven (let us say the height of Mt. Olympus, 3km) into the ocean, could they have survived? What if they had fallen into a rainforest jungle?

a) If the viscosity is too high, making the Reynolds number small, the drag will be huge and will kill our friends with a giant g-force. From a second floor window (20 m), the Catholics will hit with velocity $\sqrt{2gh} = 20\,\mathrm{m/s}$. Assuming the Catholics land feet first (instead of belly flopping), the length scale is the radius of their cross-section, say 25 cm. So the Reynolds number is $Rv/\nu = 25 \cdot 2000/\nu = 50000/\nu$. So for say $\nu < 500$ we'll get nice turbulence.

b) For any reasonable viscosity, the flow is turbulent; when you've displaced a mass of manure equal to your mass, you'll lose half your momentum. So a thickness of manure about equal to a person height (2 m) would halve your velocity (manure has about the same density as a person), if you fall standing up. Since speed scales with the square root of the height, 5 m would be the speed if you fell from half a story (one-fourth of two stories). You can probably survive that with minor injuries, if you bend your legs as you hit the ground. If you belly flop, your thickness is about 0.3 m, so you would decelerate by 10 m/s in 0.3 m/s, or 0.03 s. That's 30 g's, maybe enough to rupture your organs.

c) From heaven, you will hit with terminal velocity. From the second lecture, the terminal velocity (in fetal tuck) is 60 m/s. So, if you spreadeagle, you reduce velocity a little, say a factor of 2, so that when you hit the water (feet first) at 30 m/s, you lose half your velocity (or 15 m/s) in 2 m. This makes for a deceleration of 225 m/s, about 22 g's. You might survive. If however, you fall in the air feet first or fetal tuck, your velocity may be 3 or 4 times higher; since the g force (or drag force) scales with v^2, you'll get a few hundred g's, which will kill you.

In a rainforest jungle, you will live because the mean density of branches is lower than in a true fluid, so you decelerate over a longer distance (a hundred meters or so). [However, if the density is too low, say less than a couple percent, you might be going too fast when the ground hits you; then you die.]

5. Estimate the natural frequencies of oscillation of Millikan library in the N-S and E-W directions.

Millikan has one end (ground) fixed, and one end free. The natural frequency of oscillation depends on what fraction of the building's mass is in vertical structural supports which resist bending in the given direction. For example, for oscillations in the E-W direction, the restoring forces are due to the N-S walls. The east and west walls as well as the floors, ceilings, books and furniture are simply inertia, contributing nothing to the restoring force.

If the walls were structural (e.g. solid steel plates), the natural frequency for oscillation in the x-direction would be

$$\omega^2 \sim 2\frac{Bl_x^3 l_y}{l_z^3 m}$$

where m is the mass of the library, l_y is the wall thickness, l_x is the extent of the wall along the oscillation direction, and the factor of 2 arises because there are two walls. Millikan has $l_{N-S} \approx 10$m, $l_{E-W} \approx$15m and $l_z \approx$50m. If the walls were solid steel, ($\rho_w \approx 7\text{gcm}^{-3}, \text{B} \approx 10^{12}$) and as thick as the external pillars, $l_y \approx 50$cm, then if the mass of the floors were comparable to that in a pair of walls, and the mass of books were again comparable, then we would have $m \approx 6\rho_w l_x l_y l_z$, so

$$\omega^2 \approx \frac{Bl_x^2}{l_z^4 3\rho_w}$$

and thus the natural period of oscillation would be

$$P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3\rho_w}{B}}\frac{l_z}{l_x}l_z,$$

which is about 0.7s for the N-S direction and about 0.5s for the E-W direction.

In fact, not all of the walls is structural (plaster, glass windows, granite facing, etc.); the structural members are braced pillars (visible outside the first floor), so the resonant periods are actually a bit longer. Notice that to the extent that all tall buildings have similar shapes (l_x/l_z), $P \propto l_z$, and we have roughly derived the civil engineer's rule of thumb: one second per 10 stories.