Problem S1.[5 points] *To lift yourself out of the swimming pool, you must overcome the forces of both surface tension and gravity.*
  a) Estimate the ratio of these two forces.
  b) Repeat the calculation for a fly in the pool.

For a cubical object of size L, the surface tension force is $F_\gamma \sim 4L\gamma$, since surface tension acts over a circumference $\sim 4L$. The gravitational force is $F_\mathrm{g} \sim \rho g L^3$. Their ratio is

$$R \equiv \frac{F_\gamma}{F_\mathrm{g}} \sim \frac{4\gamma}{\rho g L^2}. \tag{1}$$

For water, $\gamma \sim 70\ \mathrm{dyne\ cm}^{-1}$. For both humans and flies, $\rho \sim 1\ \mathrm{g\ cm}^{-3}$. Scaling (1) relative to $L = 1$ cm, we have

$$R \sim \frac{4 \times 70\ \mathrm{dyne\ cm}^{-1}}{1\ \mathrm{g\ cm}^{-3} \times 1000\ \mathrm{cm\ s}^{-2}}\, L^{-2} \sim 0.3 \times \left(\frac{1\ \mathrm{cm}}{L}\right)^2. \tag{2}$$

a) For a human, $L \sim 50$ cm, so $\boxed{R \sim 10^{-4}}$.

b) For a fly, $L \sim 0.5$ cm, so $\boxed{R \sim 1}$. Surface tension is a large effect for a fly. If wetted, the fly might drown; if not wetted, the fly can float on water.

Problem S2.[5 points] *How does the incident acoustic energy flux of a whisper compare to the incident visible electromagnetic energy flux from a 5'th magnitude (faint naked eye) star? Useful facts: i) a whisper is about 20 decibels where the sound level in decibels is related to the flux of acoustic energy $\mathcal{P}$ by $10\log(\mathcal{P}/10^{-12}\mathrm{watt\ m}^{-2})$; ii) the sun would appear as a 5'th magnitude star if it were at a distance of $10\,pc \approx 3 \times 10^{19}$ cm.*

At $R = 10\ \mathrm{pc} = 3 \cdot 10^{17}$ m, the energy flux from the sun is

$$\mathcal{P}_\mathrm{star} \sim \frac{L_\mathrm{sun}}{4\pi R^2} \sim \frac{4 \cdot 10^{26}\ \mathrm{W}}{4 \times 3 \times (3 \cdot 10^{17})^2\ \mathrm{m}^2} \sim 4 \cdot 10^{-10}\ \mathrm{W\ m}^{-2}\ , \tag{1}$$

where we used $L_\mathrm{sun} \sim 4 \cdot 10^{26}$ W from Purcell's sheet. Perhaps half the energy is visible light, so

$$\boxed{\mathcal{P}_\mathrm{star} \sim 2 \cdot 10^{-10}\ \mathrm{W\ m}^{-2}}\ . \tag{2}$$

20 dB corresponds to an acoustic energy flux of $\boxed{\mathcal{P}_\mathrm{sound} \sim 10^{-10}\ \mathrm{W\ m}^{-2}}$, so the electromagnetic flux is slightly greater.

Problem S3.[5 points] *The electron is the lightest member of the family of 3 leptons which also includes the muon and tauon. Suppose the world were made up of muons and nucleons instead of electrons and nucleons. Describe one aspect in which it would differ from the world as we know it. Note: $m_\mu \approx 207 m_e$.*

Let m_ℓ be the mass of the standard lepton. The Rydberg energy scales as $\mathrm{Ry}_\infty \sim m_\ell(\alpha c)^2$, since $\alpha \equiv e^2/\hbar c$ is the dimensionless velocity of a hydrogen electron, and α does not change when

m_ℓ changes. Replacing m_e by m_μ will increase the Rydberg by a factor of $m_\mu/m_e \sim 200$, to $200 \times 14\,\mathrm{eV} \sim 3\,\mathrm{keV}$.

The Bohr radius is $\alpha^{-2} r_0$, where $r_0 \equiv e^2/m_\ell c^2$ is the classical lepton radius. So the Bohr radius will decrease by a factor of ~ 200, to 0.25 pm. Densities will increase by $200^3 \sim 10^7$, and bulk moduli (cohesive energy densities) will increase by $200^4 \sim 10^9$.

Let M be the mass of a molecule (typically $M \sim 10^4 m_e$). The ratio of vibrational to electron transition energies will increase, from $(m_e/M)^{1/2} \sim 0.01$, to $(m_\mu/M)^{1/2} \sim 200^{1/2} \times 0.01 \sim 0.15$. Similarly, the ratio of rotational to electronic transitions will increase from $\sim 10^{-4}$ to $\sim 2 \cdot 10^{-3}$.

Other effects are possible.

Problem S4.[5 points] *Estimate the thickness a one meter by one meter glass window pane must have in order to withstand winds of* $100\,km\ h^{-1}$.

Two kinds of membrane can form a barrier against the wind. The first kind is an object thick enough to resist bending (windows). The second kind is a thin sheet, thin enough to bend in the wind, but thick enough so that its tensile strength is large enough to resist tearing (parachutes and solar sails). We consider windows first, since that is what the question asks about.

A window is a two-dimensional strut; therefore our solution follows the bending-strut example discussed in class. Let $l \sim 1$ m be the characteristic size of the window, and t be its thickness. The wind will deform the window (produce strain); when the strain reaches the breaking strain the window will break.

The elastic energy is

$$E \sim \epsilon^2 l^2 t \mathcal{M}, \tag{1}$$

where $l^2 t$ is the volume of the pane, and $\mathcal{M}$ is its elastic modulus. The force is

$$F_{\text{elastic}} \sim \frac{\partial E}{\partial(\Delta y)} \sim \epsilon l^2 t \mathcal{M} \frac{\partial \epsilon}{\partial(\Delta y)}, \tag{2}$$

where Δy is the characteristic deflection of the window. We need to find an expression for $\partial\epsilon/\partial(\Delta y)$.

Let the radius of curvature of the window be R. From Figure 1 (dropping the constants), we see that the strain is

$$\epsilon \sim \frac{\Delta l}{l} \sim \frac{t}{R}. \tag{3}$$

From Figure 2, we see that in terms of the maximum deflection, the radius of curvature is

$$R \sim \frac{l^2}{\Delta y}, \tag{4}$$

if we again drop the dimensionless constants. Combining (3) and (4), we have

$$\epsilon \sim \frac{t \Delta y}{l^2}, \tag{5}$$

and therefore $\partial\epsilon/\partial(\Delta y) \sim t/l^2$. When we substitute this result into (2), we obtain $F_{\text{elastic}} \sim \epsilon t^2 \mathcal{M}$.

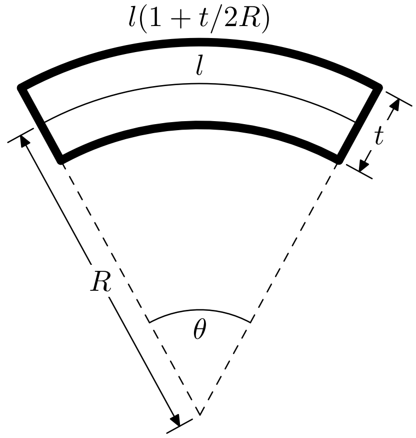

Figure 1. A piece of material (the dark lines) bent to have radius of curvature R. The thickness is t; the length of the central surface (the thin solid line) is l. The angle is $\theta = l/R$, so the length of the outer surface is $(R+t/2)\theta = l(1+t/2R)$. The strain at the outer surface is therefore $\epsilon \sim \Delta l/l = t/2R$.

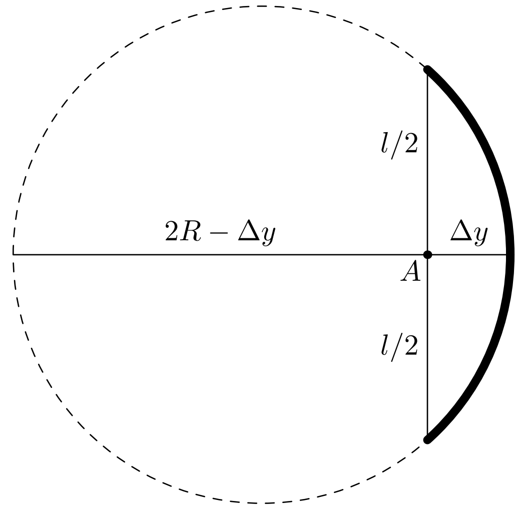

Figure 2. A radius of curvature R produces a deflection Δy in the central surface (dark line). We will use one of the power theorems of geometry to express Δy in terms of R. The point A splits the diameter into a segment of length $2R - \Delta y \approx 2R$, and one of length Δy. It splits the perpendicular chord into two segments each of length $l/2$. The power theorem says that the product of the split lengths is independent of the chord, so $2R\Delta y \approx l^2/4$. Therefore $R = l^2/8\Delta y$.

The force from the wind is $F_{\text{wind}} \sim \rho_{\text{a}} v^2 l^2$, where ρ_{a} is the density of air, and v is the wind velocity. Equating the elastic force to the wind force,

$$\rho_{\text{a}} v^2 l^2 \sim \epsilon t^2 \mathcal{M}, \tag{6}$$

and solving for the critical thickness, we have

$$t \sim l \left(\frac{\rho_{\text{a}} v^2}{\epsilon \mathcal{M}} \right)^{1/2}. \tag{7}$$

The numerator of the expression parenthesis is the ram pressure from the wind, $p \sim \rho_{\text{a}} v^2$. Here, $v \sim 100\,\text{km}\,\text{h}^{-1} \sim 3{\cdot}10^3\,\text{cm}\,\text{s}^{-1}$, so $p \sim 10^4\,\text{erg}\,\text{cm}^{-3}$. The denominator, if we allow no safety margin, is the yield stress. Taking $\mathcal{M} \sim 7{\cdot}10^{11}\,\text{erg}\,\text{cm}^{-3}$ and $\epsilon \sim 10^{-3}$, we have $\sigma_{\text{y}} \sim 7{\cdot}10^8\,\text{erg}\,\text{cm}^{-3}$. Then

$$t \sim 100\text{ cm} \times \sqrt{\frac{10^4}{7 \cdot 10^8}} \sim 0.3\text{ cm}. \tag{8}$$

To include a safety margin, we will estimate that $\boxed{t \sim 0.5\text{ cm}}$.

[Actually with $t \sim 0.3$ cm, a strain of $\epsilon \sim 10^{-3}$ corresponds to $R \sim 300$ cm (using (3)), which corresponds to $\Delta y \sim 30$ cm (using (4)). So Figure 1 is roughly to scale, and a window that bent as much would probably fall out of its frame. Therefore a real 1 m window would be significantly thicker than 0.3 cm, with $t \sim 1$ cm.]

We can express (7) as

$$\frac{t}{l} > \left(\frac{p}{\sigma_{\mathrm{y}}}\right)^{1/2}, \tag{9}$$

where $p \sim \rho_{\mathrm{a}} v^2$ is the wind pressure and $\sigma_{\mathrm{y}} = \epsilon \mathcal{M}$ is the yield stress. Among the four variables—p, σ_{y}, t and l—there are two dimensionless Pi variables (the four variables contain two dimensions: pressure and length). The Pi theorem then tells us that

$$\frac{t}{l} \sim f\left(\frac{p}{\sigma_{\mathrm{y}}}\right), \tag{10}$$

where f is a dimensionless function of a dimensionless argument. For the solution given in (9), we have $f(x) = \sqrt{x}$.

Now we will treat the solar sail, or parachute, case, which applies if the membrane is very thin. A thin-enough membrane can bend into a hemisphere without breaking. In a hemisphere of thickness t and half-circumference l, the outer surface has extra length $\Delta l \sim t$, or strain $\epsilon \sim t/l$. Therefore, for the membrane to bend, we must have $t/l < \epsilon_{\mathrm{y}}$. The requirement that the membrane bend sets a maximum thickness. The requirement that it not tear in the wind will set a minimum thickness.

In this hemispherical configuration, the membrane will break if the wind force exceeds its tensile strength; breaking will happen when $\rho_{\mathrm{a}} v^2 l^2 \sim \sigma_{\mathrm{y}} t l$, since tl is roughly the cross-sectional area of the window. Solving for the minimum thickness-to-length ratio, we have:

$$\frac{t}{l} \sim \frac{p}{\sigma_{\mathrm{y}}}, \tag{11}$$

which is of the form given in (10), with $f(x) = x$. This parachute scaling will apply to the solar sail of problem L4.

Above we found that $\rho_{\mathrm{a}} v^2/\sigma_{\mathrm{y}} \sim 1.4 \cdot 10^{-5}$. Thus (11) gives $t > 10^{-3}$ cm (provided we assume that the parachute material has the same yield stress as glass). For the membrane to be able to bend into a parachute, we require $t/l < \epsilon_{\mathrm{y}} \sim 10^{-3}$, or $t < 10^{-1}$ cm.

Problem S5.[5 points] *Consider the stability of a two dimensional, incompressible, stratified, shear flow. Let z denote the vertical coordinate and g the constant gravitational acceleration. The flow is characterized locally (in z) by the logarithmic density gradient, $d \ln \rho/dz < 0$, and the vertical shear of the horizontal (z) velocity, du_x/dz.*

a) Provide a dimensionless parameter whose value determines the stability of the flow.

b) What is the physical significance of this parameter?

a) The dimensions:

$d \ln \rho/dz$	$[L]^{-1}$
du_x/dz	$[T]^{-1}$
g	$[L][T]^{-2}$

A dimensionless combination of these expressions is

$$\Pi_1 = \boxed{g\left(\frac{d\ln\rho}{dz}\right) \Big/ \left(\frac{du_x}{dz}\right)^2}, \tag{1}$$

which is the Richardson number, Ri.

b) Consider a slice of ocean, with thickness $\sim \Delta z$, and work in the rest frame frame of the fluid in the center of the slice. Then the numerator of (1) is

$$N = \frac{g}{\rho}\frac{\Delta\rho}{\Delta z}, \tag{2}$$

and the denominator is

$$D = \left(\frac{\Delta u_x}{\Delta z}\right)^2. \tag{3}$$

If we multiply both N and D by $\rho(\Delta z)^2$, then we get

$$N' = g\Delta\rho\Delta z, \tag{4}$$

and

$$D' = \rho(\Delta u_x)^2. \tag{5}$$

N' is the gravitational potential energy change due to moving a unit volume of fluid from the edge of the slice to the center. D' is the corresponding kinetic energy change (since Δu_x is the velocity at the edge of the slice, and the fluid would be at rest in the center.) So Ri measures the stratification (how much the fluid resists generation of turbulence): large, negative values of Ri imply stability.

Problem S6.[8 points] *A particle of speed $v = \beta c$ moving through a medium with index of refraction n emits Cerenkov radiation if $\beta > 1/n$. The energy radiated in Cerenkov radiation at frequencies between ω_1 and ω_2, per unit length traversed by the particle, $dE(\omega_1, \omega_2)/dx$ depends on the electron charge e, speed of light c, ω_1, ω_2, n, β.*

a) *Use the Buckingham Pi theorem to find an expression for dE/dx.*

b) *Recall that at frequency ω, the index of refraction for a medium composed of atoms whose electrons are approximated as harmonic oscillators of natural frequency ω_0 is given by $n^2 = 1 + 4\pi N e^2/(\omega_0^2 - \omega_0)$, where N is the number of electrons per unit volume. For ultrarelativistic particles, the dE/dx of Cerenkov radiation is independent of β, and in a dilute medium such as air, with $n-1 \ll 1$, $dE/dx \propto N$. Find an expression for dE/dx in this case ($\beta \to 1$, $n-1 \ll 1$).*

a) Since dE/dx is proportional to $d\omega$, we have

$$\frac{d^2E(\omega)}{dx\,d\omega} = f(e^2, c, \omega, n, \beta), \tag{1}$$

where we used e^2 instead of e since e^2 has the more intuitive units. The variables n and β are already dimensionless. The remaining four variables have the following dimensions (in an *energy*, *velocity*, *length* system of units):

$$\begin{array}{ll} d^2E/dx\,d\omega & [E][V]^{-1} \\ e^2 & [E]\cdot[L] \\ c & [V] \\ \omega & [V][L]^{-1} \end{array} \tag{2}$$

There is only 1 Pi variable (there are 4 original variables and 3 dimensions), which we can take to be

$$\Pi_1 = \frac{d^2E(\omega)}{dx\,d\omega} \bigg/ \frac{e^2\omega}{c^2}. \tag{3}$$

From the Pi theorem, we have $g(\Pi_1, n, \beta) = 0$, or $\Pi_1 = f(n, \beta)$. So

$$\boxed{\frac{dE(\omega)}{dx} \sim \frac{e^2\omega}{c^2} f(n,\beta)\, d\omega}. \tag{4}$$

b) The index of refraction is

$$n^2 = 1 + \frac{4\pi N e^2}{m_e(\omega_0^2 - \omega^2)}. \tag{5}$$

For dilute media, where $n - 1 \ll 1$, this expression reduces to $n - 1 \propto N$. Since we are told that $dE/dx \propto N$ in this limit, and we are working in the ultrarelativistic limit (where β drops out), we know that $f(n, \beta) \sim n - 1$. Therefore

$$\frac{dE(\omega)}{dx} \sim \frac{e^2\omega}{c^2}(n-1)\, d\omega. \tag{6}$$

Problem L1.[10 points] *Fun with turkeys*

A mad scientist stuffs his Thanksgiving turkey with liquid oxygen, attaches a nozzle and ignites it; the resulting burn lasts several seconds. Estimate the maximum altitude the charred remains could attain

a) if the earth had no atmosphere.
b) including the effects of the earth's actual atmosphere.
c) in case (b), how does the altitude attained scale with the size of the turkey?

We will solve this using a conservation-of-energy argument. The argument to be given is wrong, but Sterl and the TAs used it when they solved it initially, so we're giving full credit for it. For the correct solution see Sterl's handwritten solution in the Interaction Room.

a) Combustion of dry turkey releases $\mathcal{E} \sim 6\text{kcal}\,\text{g}^{-1}$ (taking turkey to be a mixture of carbohydrate, protein and fat). We assume that a fraction of the mass, $f \sim 0.2$, is combustible (the rest is water), and that only a fraction $\epsilon \sim 0.3$ of the combustion energy is converted into gravitational potential energy via thrust. Then $mgh \sim f\epsilon m\mathcal{E}$, where m is the mass of the turkey; therefore

$$h \sim f\epsilon\mathcal{E}/g \sim \frac{0.2 \times 0.3 \times 2.5 \cdot 10^{11}\ \text{erg g}^{-1}}{1000\ \text{cm s}^{-2}} \sim 1.5 \cdot 10^7\ \text{cm} = \boxed{150\ \text{km}}. \tag{1}$$

This part of the solution is fine; Sterl's solution provides a way to estimate ϵ.

b) The turkey is launched with a supersonic velocity: $v_0 \sim \sqrt{gh} \sim 10^5\ \text{cm s}^{-1}$. When the turkey has displaced a mass of air equal to its own mass, it will have lost $\sim 1/2$ its momentum. Picture a cubical turkey, of side length L. To displace a mass of air $m \sim \rho_\text{t} L^3$, where ρ_t is the density of turkey, it must travel a distance d given by

$$\rho_\text{a} L^2 d \sim \rho_\text{t} L^3. \tag{2}$$

Therefore

$$d \sim L\,\frac{\rho_\text{t}}{\rho_\text{a}}. \tag{3}$$

If the turkey has mass $m \sim 10\,\text{kg}$, and density $\rho_\text{t} \sim 1\,\text{g}\,\text{cm}^{-3}$, then its size is $L \sim (m/\rho_\text{t})^{1/3} \sim 20\,\text{cm}$, so

$$d \sim 20 \text{ cm} \times 1000 = 0.2 \text{ km}. \tag{4}$$

It takes many such heights to slow the supersonic turkey down to a reasonable speed, slow enough that gravity can reverse its velocity. We will assume it takes 5 such heights. Then $h \sim 5d \sim \boxed{1 \text{ km}}$.

This method is only correct for an extremely fast burn, less than ~ 0.1 s. With a slower burn, drag will have time to slow the turkey down during the powered portion of the flight, to a terminal velocity determined by equating the thrust to the air drag. The coasting part of the flight will be shorter, and most of the height will be attained during the powered part of the flight. See the handwritten solution for the proper treatment of these effects.

c) From (3), we see that $d \propto L$, so the height attained will be $\boxed{\propto L}$ as well, until $h \sim O(6000\,\text{km})$, and g becomes significantly smaller. Again, this result will be different if we include the powered part of the flight.

Problem L2.[10 points] *Toaster physics*

An American (120V) toaster uses 400W in each bread slot. Each slot is wound with 300cm of metal ribbon 10^{-2} cm *thick, which serves as the heating element.*

a) *Estimate the width of the metal ribbon.*
b) *Estimate (using fundamental physics and the numbers provided) the temperature of the ribbon when the toaster is on.*
c) *How long does the ribbon take to reach this temperature after the toaster is turned on?*

a) Using $P = V^2/R$, with $P = 400$ W and $V = 120$ V, we have $R \sim (120)^2/400\,\Omega \sim 35\,\Omega$. The resistance is $R = \rho l/wt$, where ρ is the resistivity, l is the length, w is the width, and t is the thickness of the ribbon. For copper at room temperature, Purcell's sheet quotes $\rho \sim 2\cdot 10^{-6}\,\Omega\cdot\text{cm}$. The ribbon is red hot, say $T \sim 1500\,\text{K}$, so we should increase ρ by a factor of 5 for the lower phonon mean free path ($\rho \propto T$, as we found in the light bulb problem). Also toaster filaments are not made of copper; they are made of some higher resistance material, say tungsten, so we should multiply ρ by 3 or so. So we will take $\rho \sim 3\cdot 10^{-5}\,\Omega\cdot\text{cm}$.

The width is given by

$$w \sim \frac{\rho l}{tR} \sim \frac{3\cdot 10^{-5}\,\Omega\cdot\text{cm} \times 300 \text{ cm}}{0.01 \text{ cm} \times 35\,\Omega} \sim \boxed{0.03 \text{ cm}}. \tag{1}$$

Sterl measured a width of 0.1 cm in an actual toaster.

b) The filament is a blackbody with surface area

$$A \sim 2(w+t)l \sim 2 \times 0.04 \text{ cm} \times 300 \text{ cm} \sim 24 \text{ cm}^2, \tag{2}$$

radiating $P = 400$ W. Since $P \sim A\sigma T^4$, we have

$$T \sim \left(\frac{P}{\sigma A}\right)^{1/4} \sim \left(\frac{400 \text{ W}}{24 \text{ cm}^2 \times 6\cdot 10^{-12} \text{ W K}^{-4} \text{ cm}^{-2}}\right)^{1/4} \sim \boxed{1500 \text{ K}}. \tag{3}$$

[The high temperature is the reason ribbons are not made out of copper, which melts at $\sim 1400\,\mathrm{K}$— or even iron, which would be marginal (at melts at $\sim 1800\,\mathrm{K}$). Tungsten, which melts at $\sim 3700\,\mathrm{K}$, is a much better choice.]

c) The thermal energy stored in the filament is $E \sim wltc_{\mathrm{p}}\Delta T$, where c_{p} is the specific heat per volume, and $\Delta T \sim 1200\,\mathrm{K}$ is the temperature change. From Purcell's sheet, $c_{\mathrm{p}} \sim 0.5\,\mathrm{cal\,cm^{-3}\,K^{-1}}$. Using (1) for the width, we have

$$E \sim 0.03 \text{ cm} \times 300 \text{ cm} \times 0.01 \text{ cm} \times 0.5 \text{ cal cm}^{-3} \text{ K}^{-1} \times 1200 \text{ K} \sim 60 \text{ cal} \sim 250 \text{ J}. \tag{4}$$

The time to generate this energy is the heating time:

$$\tau \sim \frac{E}{P} \sim \frac{250 \text{ J}}{400 \text{ W}} \sim \boxed{0.6 \text{ s}}. \tag{5}$$

Problem L3.[10 points] *The blood and guts of exercise*

You are pedaling your bicycle as hard as you can, generating 300W of mechanical power.

a) *If there were no heat flow out of your leg muscles, how long would it take for the temperature deep inside your leg to rise by $1\,\mathrm{K}$ (onset of fever)?*

b) *If the inner temperature of the leg did rise by $1\,\mathrm{K}$, what fraction of the heat generated by the leg during vigorous exercise could be carried by static thermal conduction in the leg tissue?*

c) *The excess heat is carried from the interior to the skin by the blood vessels. Estimate the volume of blood flow (in $\mathrm{cm^3\,s^{-1}}$ required to transport the heat generated by the legs during exercise.*

d) *Estimate the mass of oxygen per unit time ($\mathrm{g\,s^{-1}}$) which must be carried to the muscles to sustain the exercise aerobically.*

e) *50% of your blood volume is occupied by red blood cells. $1\,\mathrm{cm^3}$ of packed red blood cells contain 0.35 g of hemoglobin. Oxygenated hemoglobin contains 1 O_2 molecule. Use this information, combined with your answers to parts (c) and (d), to estimate the molecular weight of hemoglobin.*

f) *Your white (anaerobic) muscles store enough energy to sustain maximum output for about 30 seconds. Would there be a point to storing more energy?*

a) The human body is about 20% efficient which means that if you generate $P = 300$ W of mechanical power you will produce a further 1200 W of heat: the rate of heat production is $\dot{Q} = 1.2 \cdot 10^{10}\ \mathrm{erg\,s^{-1}}$. If none of this heat is transported away from the legs, the time, t, it would take to raise the temperature of the legs by ΔT is given by

$$t = \frac{cm\Delta T}{\dot{Q}}, \tag{1}$$

where c is the heat capacity of leg. If we take the mass of your legs to be ~ 20 kg, and their heat capacity to be that of water, then the time it takes to raise their temperature by $1\,\mathrm{K}$ is

$$t \sim \frac{4.2 \cdot 10^7 \text{ erg g}^{-1} \text{ K}^{-1} \times 2 \cdot 10^4 \text{ g} \times 1 \text{ K}}{1.2 \cdot 10^{10} \text{ erg s}^{-1}} \sim \boxed{1 \text{ min}}. \tag{2}$$

b) If we treat each leg as a cylinder of radius r and length h, the rate of heat loss, $\dot{H}$, from both legs is the flux across a thermal boundary layer of thickness r, times the surface area:

$$\frac{dH}{dt} \sim K\frac{\Delta T}{r} \times 2 \times 2\pi rh = 4\pi K\Delta Th, \tag{3}$$

where K is the thermal conductivity of thigh. Using Purcell's sheet, we take the thermal conductivity to be

$$K \sim 10^{-2}\,\mathrm{cal\,s^{-1}\,cm^{-1}\,K^{-1}} \sim 4.2 \cdot 10^{5}\,\mathrm{erg\,s^{-1}\,cm^{-1}\,K^{-1}}. \tag{4}$$

Then (3) becomes

$$\dot{H} \sim 4\pi \times 4.2 \cdot 10^{5}\,\mathrm{erg\,s^{-1}\,cm^{-1}\,K^{-1}} \times 1\,\mathrm{K} \times 50\,\mathrm{cm} \sim 2.5 \cdot 10^{8}\,\mathrm{erg\,s^{-1}} = \boxed{25\,\mathrm{W}}. \tag{5}$$

Only about 2% of the heat generated could be lost by thermal conduction.

c) If the mass flow rate of blood is $\dot{M}$, then the rate at which blood transports heat is

$$\dot{Q} = c\dot{M}\Delta T, \tag{6}$$

where c is the heat capacity of blood. If we take the heat capacity of blood to be that of water, we find that the flow rate is

$$\dot{M} = \frac{1.2 \cdot 10^{10}\,\mathrm{erg\,s^{-1}}}{4.2 \cdot 10^{7}\,\mathrm{erg\,g^{-1}\,K^{-1}} \times 1\,\mathrm{K}} \sim 300\,\mathrm{g\,s^{-1}}. \tag{7}$$

So the volume flow rate of blood is about $\boxed{300\,\mathrm{cm^3\,s^{-1}}}$. [Note: the stroke volume of the human heart is about 70 cm^3, so this volume flow rate corresponds to a heartrate of about 250 beats per minute.]

d) The chemical reaction for respiration of carbohydrate is

$$\mathrm{CH_2O + O_2 \rightarrow CO_2 + H_2O + \Delta E}; \qquad \Delta\mathrm{E} \sim 4\,\mathrm{kcal\,g^{-1}\ of\ CH_2O} \tag{8}$$

The molecular mass of CH_2O is 30 and that of O_2 is 32 so the mass ratio of carbohydrate to oxygen is about 1:1. If $\dot{M}$ is the rate of oxygen consumption by mass, then

$$(\dot{Q} + P) = \dot{M}\Delta E, \tag{9}$$

so

$$\dot{M} \sim \frac{1.2 \cdot 10^{10}\,\mathrm{erg\,s^{-1}} + 0.3 \cdot 10^{10}\,\mathrm{erg\,s^{-1}}}{1.7 \cdot 10^{11}\,\mathrm{erg\,g^{-1}}} \sim \boxed{0.1\,\mathrm{g\,s^{-1}}}. \tag{10}$$

[Note: $0.1\,\mathrm{g s^{-1}}$ of O_2 is $100\,\mathrm{cm^3 s^{-1}}$. Assuming a body mass of 70kg, a $100\,\mathrm{cm}^3$ oxygen consumption rate means an oxygen consumption per unit body mass of $\dot{V}_{O_2} \sim 86\,\mathrm{cm^3\,min^{-1}\,kg^{-1}}$. The highest directly measured value of $\dot{V}_{O_2}$ is 93 $\mathrm{cm^3\,min^{-1}\,kg^{-1}}$, which belonged to a Scandanavian cross country skier.]

e) In parts (c) and (d) it was found that 300 cm^3 of blood carries 0.1 g of O_2. Given that blood is 50% red blood cells by volume and that there is 0.35 g of hemoglobin in every 1 cm^3 of red cells this means that $150\,\mathrm{cm^3} \times 0.35\,\mathrm{g\,cm^{-3}} \sim 50$ g of hemoglobin is needed to carry 0.1 g of O_2. Thus the molecular mass of hemoglobin is about $(50/0.1) \times \mu_{O_2} = 500 \times 32\,\mathrm{amu} = \boxed{16{,}000\,\mathrm{amu}}$.

[Normal human hemoglobin consists of four chains, each of which is capable of carrying one O_2 molecule. There are two α-chains, each with mass 15,126 amu, and two β-chains, each with mass 15,867 amu.]

f) The results above suggest that the ability of the blood to transport oxygen to the muscles and its ability to transport heat away from the muscles are optimized. The implication is that periods of anaerobic respiration (which produces energy without consuming oxygen) much beyond 30 seconds would lead to a $\boxed{\text{dangerous rise in body temperature}}$.

Problem L4.[10 points] *Interplanetary Sailing Using Solar Radiation*

a) *A circular sail composed of aluminum foil is attached to a spacecraft of equal mass in a manner similar to a parachute (i.e. the spacecraft hangs in the center of the sail on the sunward side, suspended by threads which extend from the circumference of the sail). What is the maximum thickness of the aluminum foil which would permit sailing in the radial direction away from the Sun?*

b) *What is the maximum radius that a solar sail composed of a single sheet of aluminum could have if it is to avoid tearing? Consider the system described in (a).*

c) *Estimate the minimum thickness of aluminum foil sufficient to reflect most solar radiation.*

a) The mass of the sail is $M_{\rm sail} = \pi R^2 t\rho$, where R is the radius of the sail, t is its thickness, and ρ is its density, so the mass of the sail plus spacecraft is

$$M \sim 2\pi R^2 t\rho. \qquad (1).$$

At a distance D from the sun, the radiation pressure is $P = 2\mathcal{P}/c$, where $\mathcal{P} \sim L_{\rm sun}/4\pi D^2$ is the energy flux from the sun (the 2 is because almost all the radiation reflects off the sail, as we will see in part c). The force over the whole sail is therefore

$$F_{\rm rad} \sim \pi R^2 P \sim \frac{\pi R^2}{4\pi D^2}\frac{2L_{\rm sun}}{c} = \frac{1}{2}\left(\frac{R}{D}\right)^2 \frac{L_{\rm sun}}{c}. \qquad (2)$$

For the ship to make progress, this force must at least balance solar gravity, so

$$\frac{1}{2}\left(\frac{R}{D}\right)^2 \frac{L_{\rm sun}}{c} \sim \frac{GM_{\rm sun}M}{D^2} \sim GM_{\rm sun}2\pi t\rho\left(\frac{R}{D}\right)^2, \qquad (3)$$

where we used M from (1). Simplifying (3), we find

$$t \sim \frac{L_{\rm sun}}{4\pi\rho c G M_{\rm sun}}. \qquad (4)$$

Putting in numbers from Purcell's sheet, and guessing that $\rho \sim 3\,{\rm g\,cm^{-3}}$, we have

$$t \sim \frac{4\cdot 10^{33}\,{\rm erg\,s^{-1}}}{12\times 3\,{\rm g\,cm^{-3}}\times 3\cdot 10^{10}\,{\rm cm\,s^{-1}}\times 7\cdot 10^{-8}\,{\rm erg\,cm\,g^{-2}}\times 2\cdot 10^{33}\,{\rm g}} \sim 3\cdot 10^{-5}\,{\rm cm} = \boxed{3000\,\text{Å}}. \qquad (5)$$

b) We assume that the points where the parachute strings attach to the sail are well reinforced, so that no tears occur there. Consider a hemispherical sail. The radiation force produces a stress across the equator of the hemisphere, which has cross-sectional area $A \sim 2\pi R t$. So the stress is $\sigma \sim F_{\rm rad}/2\pi R t$ and the strain is $\epsilon \sim \sigma/\mathcal{M} \sim F_{\rm rad}/2\pi R t\mathcal{M}$, where $\mathcal{M}$ is the elastic modulus of aluminum. Using the radiation force is given by (2), we have

$$\epsilon \sim \frac{RL_{\rm sun}}{4\pi D^2 t\mathcal{M}c}. \qquad (6)$$

Solving for the radius, we have

$$R \sim \frac{4\pi D^2 t\epsilon\mathcal{M}c}{L_{\rm sun}}. \qquad (7)$$

We estimate that $\mathcal{M} \sim 10^{12}\,{\rm erg\,cm^{-3}}$, and with a reasonable safety margin, that $\epsilon \sim 10^{-3}$. The earth–sun distance will determine the largest allowed R, because the strain will be greatest there (since the spacecraft sails away from the sun), so we take $D = 1$ AU. Then, using t from (5), we have

$$R \sim \frac{12 \times 2.3 \cdot 10^{26}\ \mathrm{cm}^2 \times 3 \cdot 10^{-5}\ \mathrm{cm} \times 10^{-3} \times 10^{12}\ \mathrm{erg\ cm}^{-3} \times 3 \cdot 10^{10}\ \mathrm{cm\ s}^{-1}}{4 \cdot 10^{33}\ \mathrm{erg\ s}^{-1}} \sim 6 \cdot 10^{8}\ \mathrm{cm}. \quad (8)$$

So $\boxed{R \sim 6000\ \mathrm{km}}$, which is about the radius of the earth.

If we note that the radiation pressure is $p \sim L_{\mathrm{sun}}/(cD^2)$, then we can rewrite (7) as $t/R \sim p/(\epsilon \mathcal{M}) \sim p/\sigma_{\mathrm{y}}$. This relation is what we found in the solution to S4, when we considered the easy-to-bend, thin-membrane limit (see (11) in solution S4). The solar sail is certainly easy to bend: $t/R \sim 10^{-9} \ll \epsilon_{\mathrm{y}}$. Therefore, the strut solution (see (9) in the solution to S4) is not relevant. (The strut would only be relevant for a solar sail with $R < t/\epsilon_{\mathrm{y}} \sim 0.03$ cm; such a small sail would not make a very useful propulsion device.)

[The mass of the sail, and of the payload, is

$$M \sim \pi R^2 t \rho \sim 3 \times (6 \cdot 10^8\ \mathrm{cm})^2 \times 3 \cdot 10^{-5}\ \mathrm{cm} \times 3\ \mathrm{g\ cm}^{-3} \sim 10^{14}\ \mathrm{g} = 10^8\ \mathrm{tons}.$$

For comparison, a Nimitz-class aircraft carrier is $\sim 10^5$ tons.]

c) The free electrons in aluminum make a plasma, with plasma frequency

$$\omega_{\mathrm{p}} = \left(\frac{4\pi n_e e^2}{m_e}\right)^{1/2} = c\left(\frac{4\pi n_e e^2}{m_e c^2}\right)^{1/2} = (4\pi n_e a_0)^{1/2}\,\alpha c, \quad (9)$$

where n_e is the number density of free electrons, and we used $e^2/m_e c^2 \equiv r_0 = \alpha^2 a_0$, where r_0 is the classical electron radius. We guess that there are 2 free electrons per $(3\text{Å})^3$ cube, so $n_e \sim 10^{23}\,\mathrm{cm}^{-3}$. Then

$$\omega_{\mathrm{p}} \sim \left(4 \times 3 \times 10^{23}\ \mathrm{cm}^{-3} \times 0.5 \cdot 10^{-8}\ \mathrm{cm}\right)^{1/2} \times 0.01 \times 3 \cdot 10^{10}\ \mathrm{cm\ s}^{-1} \sim 2 \cdot 10^{16}\ \mathrm{s}^{-1}. \quad (10)$$

The index of refraction is

$$n^2 = 1 - \left(\frac{\omega_{\mathrm{p}}}{\omega}\right)^2. \quad (11)$$

When $n^2 < 0$, then n is imaginary and the wave vector $k = n\omega/c$ will be imaginary as well: the waves will be exponentially cut off in the plasma, with length constant $l \sim 1/k = c/(|n|\omega)$. What doesn't make it through the sail will be reflected.

The shortest wavelengths will travel the farthest, so we will choose the thickness so that most of the blue light is reflected. For blue light,

$$\omega \sim \frac{2\pi c}{\lambda} \sim \frac{6 \times 3 \cdot 10^{10}\ \mathrm{cm\ s}^{-1}}{5 \cdot 10^{-5}\ \mathrm{cm}} \sim 4 \cdot 10^{15}\ \mathrm{s}^{-1}. \quad (12)$$

Thus $\omega \ll \omega_{\mathrm{p}}$, so from (11) we find that $|n| \sim \omega_{\mathrm{p}}/\omega$. So

$$l \sim \frac{c}{|n|\omega} \sim \frac{c}{\omega_{\mathrm{p}}} \sim \frac{3 \cdot 10^{10}\ \mathrm{cm\ s}^{-1}}{2 \cdot 10^{16}\ \mathrm{s}^{-1}} \sim 1.5 \cdot 10^{-6}\ \mathrm{cm} = 150\ \text{Å}. \quad (13)$$

A couple length constants will reflect $\sim 1 - e^{-4} \sim 98\%$ of the light, so we estimate the required thickness as $\boxed{t \sim 300\ \text{Å}}$, which is significantly less than the maximum thickness of the foil we calculated in (5); the craft can sail.

[The usual formula for the skin depth,

$$\delta \sim \frac{c}{\sqrt{2\pi\mu\omega\sigma}}, \tag{14}$$

where σ is the conductivity, assumes that σ is real (i.e., that the wave is attenuated by eddy currents in the conductor). However, at optical frequencies, σ is almost purely imaginary in most metals, so the derivation of (14) breaks down.

To treat the general case, we start with the expression for k^2 in a conductor, from Jackson, *Classical Electrodynamics*, 1st. ed., p. 223:

$$k^2 = \mu\epsilon\frac{\omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\omega\epsilon}\right). \tag{15}$$

(This expression can be got by substituting a plane wave of frequency ω and wavevector k into Maxwell's equations.)

In the free-electron model of metals, we have

$$\frac{d\mathbf{v}}{dt} + \gamma\mathbf{v} = \frac{e\mathbf{E}}{m_e}, \tag{16}$$

where γ is the damping constant ($1/\gamma \equiv \tau$ is mean free time). The current density is given by $\mathbf{J} = n_e e\mathbf{v}$ and $\mathbf{J} = \sigma\mathbf{E}$, so $\mathbf{E} = (n_e e/\sigma)\mathbf{v}$. Therefore (16) becomes

$$\left(\frac{d}{dt} + \gamma\right)\mathbf{v} = \frac{n_e e^2}{m_e\sigma}\mathbf{v}. \tag{17}$$

Substituting in a plane wave of frequency ω, so that $\mathbf{v} \propto e^{-i\omega t}$, we find that $-i\omega + \gamma = n_e e^2/m_e\sigma$. Therefore

$$\sigma = \frac{n_e e^2}{m_e(\gamma - i\omega)} = \frac{\omega_\text{p}^2}{4\pi(\gamma - i\omega)}. \tag{18}$$

At high frequencies ($\omega \gg \gamma$), the conductivity is independent of damping, and we have

$$\sigma \sim i\frac{\omega_\text{p}^2}{4\pi\omega} \sim i\sigma_0\frac{\gamma}{\omega}, \tag{19}$$

where σ_0 is the conductivity at $\omega = 0$. The damping drops out in the high-frequency limit because an electron collides only after many cycles (so the real part of the conductivity—the dissipative part—is small).

We may estimate γ using $\gamma \sim v_\text{F}/l$, where v_F is the Fermi velocity, and l is the mean free path; typically $l \sim 100$ lattice constants in a good conductor. Taking $v_\text{F} \sim 10^8\,\text{cm}\,\text{s}^{-1}$ and $l \sim 3\cdot10^{-5}$ cm, we have $\gamma \sim 3 \cdot 10^{13}\ \text{s}^{-1}$. For $\omega \ll \gamma$, the DC conductivity is a good approximation to σ; but at optical frequencies, where $\omega \gg \gamma$, we see from (19) that σ falls as γ/ω from the DC value, and that σ is almost purely imaginary.

We can estimate γ by another method: we force the $\omega = 0$ conductivity to match the measured conductivity. From Purcell's sheet, for copper at room temperature, $\rho \sim 2 \cdot 10^{-6}\,\Omega \cdot \text{cm}$. Aluminum is a slightly worse conductor, so we'll take $\rho \sim 3 \cdot 10^{-6}\,\Omega \cdot \text{cm}$. Then the DC conductivity is

$$\sigma_0 = \rho^{-1} \sim 3 \cdot 10^5\,\Omega^{-1}\,\text{cm}^{-1} \times \frac{9 \cdot 10^{11}\,\text{cm}\,\text{s}^{-1}}{1\,\Omega^{-1}} \sim 3 \cdot 10^{17}\,\text{s}^{-1}. \tag{20}$$

Using (18) with $\omega = 0$ to solve for γ, we have

$$\gamma \sim \frac{\omega_\text{p}^2}{4\pi\sigma} \sim \frac{4 \cdot 10^{32}\,\text{s}^{-2}}{4 \times 3 \times 3 \cdot 10^{17}\,\text{s}^{-1}} \sim 10^{14}\,\text{s}^{-1}, \tag{21}$$

which is in rough agreement with the value we estimated using the collision time.

To find the skin depth, we substitute (19) into (15), to get (with $\mu = \epsilon = 1$):

$$k^2 \sim \frac{\omega^2}{c^2}\left(1 - \frac{\omega_\text{p}^2}{\omega^2}\right). \tag{22}$$

Since $n = kc/\omega$, we obtain

$$n^2 = 1 - \left(\frac{\omega_\text{p}}{\omega}\right)^2, \tag{23}$$

which is (11) again, via a much longer route.]

Problem L5.[10 points] *Bombs and torpedoes*

An explosion in an isotropic medium (like air or water) creates a spherically expanding shock front, across which the pressure increases discontinuously. A sudden overpressure of 0.1–1 atm (equivalent to 0.1-1kg weight per cm^2*) destroys living things, buildings and ships.*

a) *While the velocity of the shock is much larger than the speed of sound* c_s *of the medium through which it propagates, the shock speed* v *depends only on the density* ρ *of the medium, the instantaneous radius* R *of the shock, and the explosion energy* E*. Use the Buckingham Pi theorem to find the shock speed* v *as a function of* E*,* R *and* ρ*.*

b) *When the energy density (pressure) behind the shock drops below* ρc_s^2*, it ceases to be a strong shock, and its speed is no longer given by the* v *you found in part (a), but it instead propagates at nearly the sound speed as a (sharp-edged) wave. Find the postshock pressure as a function of* E*,* R*,* ρ *and* c_s *in this limit.*

c) *Consider two explosions of identical energy* E*, one of which occurs under water and the other in air. Compute the ratio of the distances at which the shockwave pressure falls to 1 atm,* R_{1w}/R_{1a}*, and thus show that underwater explosions are destructive to a much larger distance than the same explosions in air.*

a) We want $v = v(E, R, \rho)$. With four variables and three dimensions, we have one Pi variable: $\Pi = E/\rho R^3 v^2$. So

$$\boxed{v \sim \left(\frac{E}{\rho R^3}\right)^{1/2}}. \tag{1}$$

The postshock pressure is $\Delta p = p - \rho v^2 \sim E/R^3$.

b) In a weak shock moving at velocity $c_\mathrm{s} + \Delta v$, the pressure jump (as for sound waves), is

$$\frac{\Delta p}{\rho c_\mathrm{s}^2} \sim \frac{\Delta v}{c_\mathrm{s}}. \tag{2}$$

This result can be found by the piston argument from class, or from the jump condition of conservation of mass and momentum flux across the shock front. By the time that $\Delta p \sim \rho c_\mathrm{s}^2$, the reverse shock will have propagated through the ejected explosive. Since the internal pressure cannot drop below ambient in this stage (or an implosion would result), the weak shock must propagate as an N-wave—an overpressure region, followed by a suction region.

Let the width of the N-wave be w. The kinetic energy stored in the weak shock is

$$E \sim \rho(\Delta v)^2 4\pi R^2 w. \tag{3}$$

The thermal energy will be $O(\Delta p)^2$, since the $+$ and $-$ parts of the N-wave cancel to first order; since $\Delta p \sim \Delta v$, the thermal energy will also be $O(\Delta v)^2$, and in fact is of the same magnitude as the kinetic energy.

We will guess that w is constant and check this assumption shortly. From (3), since E is a constant, we have

$$\Delta v \propto \frac{1}{R\sqrt{w}}. \tag{4}$$

With w constant as well, we have $\Delta v \propto R^{-1}$. Since $dw/dt = 2\Delta v$ and $dR/dt \sim c_\mathrm{s}$, we have

$$\frac{dw}{dR} = \frac{dw/dt}{dR/dt} \sim \frac{\Delta v}{c_\mathrm{s}} \propto R^{-1}, \tag{5}$$

since $\dot{R} \sim c_\mathrm{s}$. So $w \propto \ln R$ (which means w is almost constant). So from (4), we have

$$\Delta v \propto \frac{1}{R\sqrt{\ln R}}. \tag{6}$$

The initial width of the shock is $R_* = (E/\rho c_\mathrm{s}^2)^{1/3}$, the radius at the strong-shock to weak-shock transition. So

$$\Delta v \sim c_\mathrm{s} \frac{R_*}{R} \frac{1}{\sqrt{\ln(R/R_*)}}, \tag{7}$$

and

$$\Delta p \sim \rho c_\mathrm{s}^2 \frac{R_*}{R} \frac{1}{\sqrt{\ln(R/R_*)}}. \tag{8}$$

c) In air $\rho c_\mathrm{s}^2 \sim 1\,\mathrm{atm}$, but in water $\rho c_\mathrm{s}^2 \sim 10^4\,\mathrm{atm}$ (10^3 times denser, and 3 times the sound speed). So in air, the postshock pressure drops at 1 atm at the strong-to-weak-shock transition:

$$R_{1\mathrm{a}} \sim R_{*\mathrm{a}} \sim \left(\frac{E}{\rho_\mathrm{a} c_\mathrm{sa}^2}\right)^{1/3}. \tag{9}$$

For $R_{1\mathrm{w}}$, we use (8) to find that

$$1\,\mathrm{atm} \sim \rho_\mathrm{w} c_\mathrm{sw}^2 \frac{R_{*\mathrm{w}}}{R_{1\mathrm{w}}} \frac{1}{\sqrt{\ln(R_{1\mathrm{w}}/R_* w)}} \sim \rho_\mathrm{w} c_\mathrm{sw}^2 \left(\frac{E}{\rho_\mathrm{w} c_\mathrm{sw}^2}\right)^{1/3} \frac{1}{R_{1\mathrm{w}}} \frac{1}{\sqrt{\ln(R_{1\mathrm{w}}/R_* w)}}. \tag{10}$$

Therefore, since $1\,\text{atm} \sim \rho_{\rm a} c_{\rm sa}^2$, we have

$$R_{\rm 1w} \sim \frac{\rho_{\rm w} c_{\rm sw}^2}{\rho_{\rm a} c_{\rm sa}^2} \left(\frac{E}{\rho_{\rm w} c_{\rm sw}^2}\right)^{1/3} \frac{1}{\sqrt{\ln(R_{\rm 1w}/R_* w)}}. \tag{11}$$

This equation, with $R_{*\rm w}$ replaced by the blast radius a_0, was first derived by Bethe and Kirkwood in their wartime study of underwater explosions (see *Shock and Detonation Waves*, by John G. Kirkwood (New York: Gordon and Breach, 1967), p. 30, eq. 5.3.)

Combining (9) and (11), we find that

$$\frac{R_{\rm 1w}}{R_{\rm 1a}} \sim \left(\frac{\rho_w c_w^2}{\rho_a c_a^2}\right)^{2/3} \frac{1}{\sqrt{\ln(R_{\rm 1w}/R_* w)}} \sim \left(10^4\right)^{2/3} \times \frac{1}{\sqrt{\ln 10^4}} \sim \boxed{100}. \tag{12}$$

Therefore are destructive for a far longer distance in water than they are in air.