Solutions to Problem Set 2

1. If the electrons in one raindrop could be removed from the Earth without removing the protons, by how much would the potential of the Earth be increased?

   Potential is $\Delta\phi = \frac{Q_e}{R_\oplus}$. The number of electrons in a raindrop is

$$N \approx \frac{1\text{cm}^3}{\text{drop}} \times \frac{1\text{g}}{\text{cm}^3} \times \frac{3\times 10^{22} H_2O}{\text{g}} \times \frac{10e^-}{H_2O} \approx 3\times 10^{23}.$$

   The charge of an electron is 4.8×10^{-10}esu and the radius of the earth is 6×10^8cm, so the potential increase is

$$\Delta\phi \approx \frac{3\times 10^{23}\times 4.8\times 10^{-10}\text{esu}}{6\times 10^8\text{cm}} \approx 2.4\times 10^5 \frac{\text{esu}}{\text{cm}} \approx 7\times 10^7\text{volts}.$$

2. Relate the magnetic field strength of a permanent iron magnet to the value of the Bohr magneton. Include a numerical value as part of your answer.

   In a ferromagnet at room temperature (well below Curie temperature), all spins in a domain are aligned. In a really good magnet, there is only one domain. Hence inside a magnet, or just outside a really good one, $B = 4\pi M = 4\pi n\mu_B$, where $n = \rho_{Fe}/m_{Fe}$ is the density of iron atoms. Note that adding the fields of dipoles leads to an internal field strength independent of the size of the system. With $\rho_{Fe} = 8\,\text{g cm}^{-3}$, $m_{Fe} = 56m_p$, and $\mu_B = e\hbar/(2m_e c)$, we find $B = 10^4$ G, the well-known characteristic maximum field strength of good ferromagnets. Note that in terms of fundamental constants, $B \sim \mu_B a_B^{-3} \sim e\alpha a_B^{-2}$, where α is the fine structure constant, and a_B is the Bohr radius (atomic spacing).

3. Earth's magnetic field
   a) Estimate the ohmic decay time for the earth's magnetic field.
   b) What does your answer in a) suggest about the origin of this magnetic field?

   a) The ohmic decay time is given by $t_\Omega \sim (4\pi\sigma R_c^2)/c^2$, where $\sigma \approx 10^{16}\,\text{s}^{-1}$ is the electrical conductivity of molten iron and $R_c \approx 3\times 10^8$ cm is the radius of the earth's liquid core. In estimating σ for liquid iron in the core, we have decreased by 0.1 the value quoted in the class notes for solid iron at 300 K. With these values, $t_\Omega \approx 3\times 10^5$ y.
   b) Since t_Ω is much smaller than the age of the earth, the field must be maintained by dynamo action. That is, appropriate differential motions of the electrically conducting fluid in the earth's core must transfer kinetic energy to magnetic energy through the stretching of magnetic field lines.

4. Of all the elements, only He, and to a lesser extent H_2 show interesting quantum effects in their liquid state. Give an order-of-magnitude calculation to explain why other liquids (including other noble gases besides He) don't show quantum effects, and are well described

by classical (e.g. hard sphere) models. [Hint: some ingredients in your calculations will be binding energies, melting temperatures and the uncertainty principle]

The momentum of an atom of mass m at temperature T is of order $p \sim \sqrt{3mkT}$. The de Broglie wavelength $\lambda \sim \frac{h}{p}$. In a liquid, the interatomic spacing is about the size of an atom a. When the de Broglie wavelength $\lambda \ll a$, atoms will collide as classical billiard balls. When $\lambda \gg a$, the atoms will interact coherently as quantum systems, and one might observe non-classical behaviour in the liquid. Notice that $\lambda \propto (mT)^{-1/2}$, so quantum behaviour is most likely to be observed for light atoms at low temperatures. The temperature can't be too low, however, or the material will solidify. So for a given material, the lowest temperature of interest is the melting temperature, T_m. As described in class, at low pressures $kT_m \sim U_b/10$, where U_b is the binding energy per atom in the solid. Therefore we would expect the lowest melting temperatures among the most weakly bound solids: i.e. those bound with only van der Waals forces —e.g. the noble gasses, and molecules with no dipole moment. Looking up heats of sublimation in our favorite handbook, and noting that to within 20%, 1kJ/mol $= 10^{-2}$eV/atom $= k(100\,\mathrm{K})$, we find that this is indeed so:

Substance	$U_b/10k$(K)	T_m(K)	$\lambda = h/\sqrt{3mkT_m}$(Å)
H_2	10	14	5
He	1	—	13
Ne	20	24	1
Ar	75	84	0.4
Kr	100	116	0.26
Fe	4000	1810	0.08
C	8300	4123	0.12

Since atoms are all a few Å across (the inter-atomic spacing in both liquid He and liquid H_2 is 3.5Å), we see from the table that only for helium, and perhaps hydrogen are the de Broglie wavelengths in the liquid larger than the atoms. All other substances are too massive, and too tightly bound (hence solidify at too high a temperature) to be likely to show quantum effects. At sufficiently low temperatures $\ll T_m$, one can observe quantum effects in the sticking and hopping of gas vapor on solid surfaces for heavier materials, but these are much less spectacular than the quantum liquid effects. [NB the quantum collective effects prevent He from solidifying at low pressures, though at high pressure it does solidify at $\sim 2\,\mathrm{K}$, about as predicted from the $U_b/10$ rule. Of course spin and statistics affect the quantum states, so the phase diagram is different for He^3 and He^4:]

5. Bathyspheres are spherical vessels designed to withstand the pressure at great depths in the ocean.

   a) How thick must the wall of a 2 meter radius bathysphere be in order for it to safely travel to the deepest parts of the ocean, depth 10 kilometers?

The minimum thickness is determined by requiring the stress induced in the vessel be less than the yield stress. We adopt a safety factor of 2. The stress, σ, in a spherical shell of radius R and thickness ΔR is related to the external water pressure, p, by

$$\sigma = \frac{pR}{2\Delta R}.$$

At 10km depth, the water pressure is $p = \rho g h \sim 10^9$dyne/cm^2. For the best quality Cr-Mo steel, the yield stress $\sigma_Y \sim 10^{10}$dyne/cm^2. Thus $\sigma = 2\sigma_Y$ for $\Delta R/R = 0.1$, or $\Delta R = 20$cm. For details regarding bathyspheres consult "The Deep Submersible", by R. Terry. The record setting vessels had $\Delta R/R \approx 0.15$.

b) Would an empty bathysphere float?

Assuming the gap is filled with air or vacuum, the mean density is about $3\Delta R/R \times \rho_{\text{steel}} \sim$ 2.4g/cc, so it would sink.