# Estimating light bending using order-of-magnitude physics

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**Abstract.** I estimate the bending of light by the sun, showing how to use dimensional analysis and order-of-magnitude physics in A-level physics.

Contents

Rocks, birds, and people feel the effect of gravity. So why not light? The analysis of that question is a triumph of Einstein's theory of general relativity. I can calculate how much gravity bends light by solving the equations of general relativity:

$$\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\alpha}} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = 0. \tag{1}$$

This notation is really shorthand for ten equations, each a partial differential equation; the set is rich in mathematical interest but is a nightmare to solve. The equations are numerous – that's one problem – but worse, they are not linear. So the standard trick, which is to guess a type of solution and form new solutions by combining the basic types, does not work. You can spend a decade learning advanced mathematics to solve the equations exactly. Or you can accept the great principle of analysis: When the going gets tough, lower your standards. That's my plan: If I give up some accuracy, I can explain light bending using mathematics and physics you (and I!) already know and in less than 10,000 pages.

## 1 Dimensional analysis

Dimensional analysis is the first method I try on new problems. It makes me think about the physics and may give an idea of the size of the effect. The experience from doing the dimensional analysis suggests how to do analyses with more physics in them.

I'll study a concrete problem: How much bending does the sun produce? This problem was one of the historical tests of general relativity, so by choosing it I can compare my numerical predictions with the measured values.

### *1.1 Finding parameters*

The first step is to decide what physical parameters can the bending angle depend on. An unlabelled diagram (Figure 1) prods me into thinking of labels, many of which are parameters of the problem.

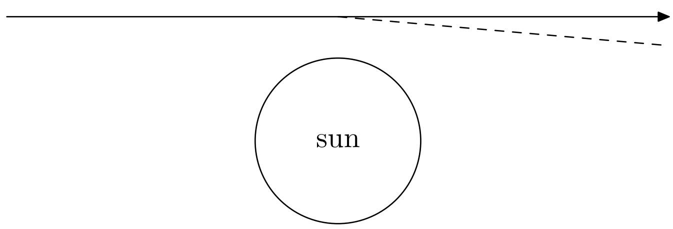


**Figure 1.** *Light being bent by an object such as the sun. The lack of labels is intentional: As I label it, I think of parameters to include in the dimensional analysis.*

I often forget to include the quantity I'm solving for – in this case, the angle, $\theta$ – but I won't this time. The mass of the sun, $m$, has to affect the angle: Black holes greatly deflect light, probably because of their huge mass. But a faraway sun or black hole cannot much affect the path (near the earth light seems to travel straight, in spite of black holes all over the universe); so $r$, the distance from the centre, is a relevant parameter. The phrase 'distance from the centre' is ambiguous, since the light is at various distances from the centre. I'll let $r$ be the distance of closest approach. I somehow need to tell the dimensional analysis that gravity does the bending; the parameters so far do not mention any physical forces. So I also include Newton's gravitational constant, $G$, in the list. Figure 2 shows the labelled diagram.

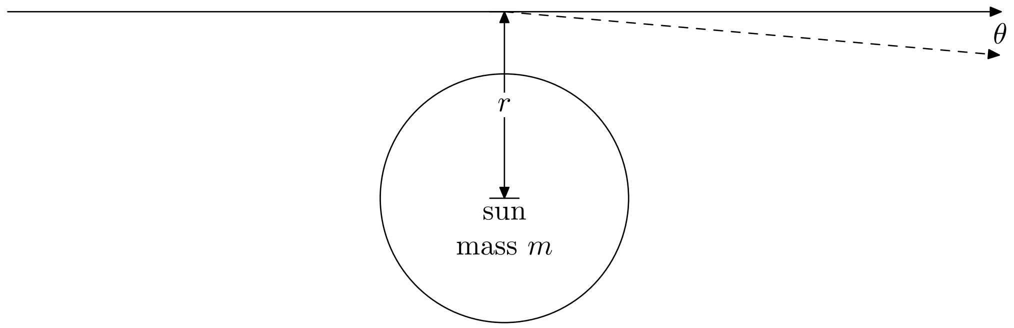


**Figure 2.** *The diagram of Figure 1 with labels indicating important physical parameters.*

*1.2 Dimensionless groups*

What are the dimensionless groups? One parameter, $\theta$, is an angle, which is already dimensionless. Can $G$, $m$, and $r$, form a second dimensionless group? Not likely: The three variables contain three independent dimensions (mass, length, and time), so there are $3 - 3$, or zero, dimensionless groups. Alas! I must have forgotten a crucial parameter. Don't take my word for it: Check for yourself. Here is a table of the parameters and their dimensions:

| *Parameter* | *Meaning* | *Dimensions* |
|---|---|---|
| $\theta$ | angle | – |
| $m$ | mass of sun | M |
| $G$ | Newton's constant | $\mathrm{L}^3\mathrm{T}^{-2}\mathrm{M}^{-1}$ |
| $r$ | distance from centre of sun | L |

where, as you might suspect, L, M, and T represent the dimensions of length, mass, and time, respectively. Can you combine the last three parameters into a dimensionless group? For example, what are the dimensions of $Gm/r$? Or of $m/r^2$?

I want a second dimensionless group because otherwise my analysis seems like nonsense. Any physical solution can be written in dimensionless form; this idea is the foundation of dimensional analysis. With only one dimensionless group, $\theta$, I have to conclude that $\theta$ depends on no variables at all:

$$\theta = \text{function of other dimensionless groups},$$

but there are no other dimensionless groups, so

$$\theta = \text{constant}.$$

This conclusion is crazy! The angle must depend on at least one of $m$ and $r$. My physical picture might be confused, but it's not so confused that neither variable is relevant. So I need to make another dimensionless group on which $\theta$ can depend. Therefore, I return to Step 1.

*1.3 Finding parameters, again!*

What physics have I neglected? Free associating often suggests the missing parameter. Unlike rocks, light is difficult to deflect, otherwise humanity would not have waited until the 1800s to study the deflection, whereas the path of rocks was studied at least as far back as Aristotle and probably for millions of years beforehand. Light travels much faster than rocks, which may explain why light is so difficult to deflect: The gravitational field 'gets hold of it' only for a short time. But none of my parameters distinguish between light and rocks. Ah, hah! I should include $c$, the speed of light. It introduces the fact that I'm studying light, and it does so with a useful distinguishing parameter, the speed. The list of parameters becomes $\theta$, $G$, $m$, $r$, and $c$.

### *1.4 Dimensionless groups, again!*

With four variables ($G$, $m$, $r$, and $c$) composed of three dimensions, I expect one dimensionless group ($\theta$ is already a group). Here is the latest table of parameters and dimensions:

| *Parameter* | *Meaning* | *Dimensions* |
|---|---|---|
| $\theta$ | angle | – |
| $m$ | mass of sun | M |
| $G$ | Newton's constant | $L^3T^{-2}M^{-1}$ |
| $r$ | distance from centre of sun | L |
| $c$ | speed of light | $LT^{-1}$ |

Length is strewn all over the parameters (it's in $G$, $r$, and $c$). Mass, however, appears in only $G$ and $m$, so I already know I need a combination such as $Gm$ to cancel out mass. Time also appears in only two parameters: $G$ and $c$. To cancel out time, I need to form $Gm/c^2$. This combination has one length in it, so a dimensionless group is $Gm/rc^2$.

### *1.5 Drawing conclusions*

The most general relation between the two dimensionless groups is

$$\theta = f\left(\frac{Gm}{rc^2}\right). \tag{2}$$

Dimensional analysis cannot tell me the correct function $f$. A simple guess, which I am never too shy to try, is that $f$ is the identity function. Then the bending angle is

$$\theta = \frac{Gm}{rc^2}.$$

More likely, there is some dimensionless constant in $f$, so

$$\theta = 7\frac{Gm}{rc^2}$$

or

$$\theta = 0.3\frac{Gm}{rc^2}$$

or a similar relation, which can be summarised as

$$\theta \sim \frac{Gm}{rc^2}. \tag{3}$$

To check this relation, I need some physics in the analysis.

## 2 Physical analysis

The guess (3) seems plausible. At least, it's more plausible than other candidates consistent with the honest result (2). For example, I could have guessed that $f(x) = 1/x$, giving

$$\theta = \frac{rc^2}{Gm}. \tag{4}$$

But this relation cannot be right. The distance, $r$, is in the numerator, so faraway black holes bend light more than nearby ones! The mass, $m$, is in the denominator, so a grain of dust bends light more than a sun! Seeing the speed of light, $c$, in the numerator provides no comfort either: It means that fast 'objects', like light, deflect more than slow ones. These considerations, which show why (4) is unreasonable, show why (3) is reasonable.

### *2.1 Interpreting the dimensionless group*

Whatever the exact result for the bending angle, dimensional analysis has told me a useful physical quantity: $Gm/rc^2$. It measures the strength of the gravitational field, as indicated by the $Gm$ in the numerator. The combination $Gm$ itself cannot measure the strength because it is not dimensionless: The strength, if measured by $Gm$, would depend on the system of units I used.

At the surface of the Earth, the strength is

$$\frac{Gm}{rc^2} \sim \frac{6.7 \cdot 10^{-11}\,\mathrm{m}^3\,\mathrm{s}^{-2}\,\mathrm{kg}^{-1} \times 6.0 \cdot 10^{24}\,\mathrm{kg}}{6.4 \cdot 10^6\,\mathrm{m} \times 3.0 \cdot 10^8\,\mathrm{m}\,\mathrm{s}^{-1} \times 3.0 \cdot 10^8\,\mathrm{m}\,\mathrm{s}^{-1}} \sim 10^{-9}.$$

This miniscule value is probably also the bending angle (in radians). So if physicists want to show that light bends, they had better look beyond the earth! I know this thanks to another piece of dimensional analysis, which I'll quote: A telescope with mirror of diameter $d$ can resolve angles roughly as small as $\lambda/d$, where $\lambda$ is the wavelength of light. One way to measure the bending of light is to measure the change in position of the stars. A lens that could resolve an angle of $10^{-9}$ has a diameter of at least

$$d \sim \lambda/\theta \sim \frac{0.5 \cdot 10^{-6}\,\mathrm{m}}{10^{-9}} \sim 500\,\mathrm{m}.$$

Large lenses warp and crack; one of the largest lenses made is 6 m. So there's no chance of detecting an angle of $10^{-9}$, at least with optical telescopes.

So physicists searched for another source of light bending. In the solar system, the largest mass is the sun. At the surface of the sun, the field strength is

$$\frac{Gm}{rc^2} \sim \frac{6.7 \cdot 10^{-11}\,\mathrm{m}^3\,\mathrm{s}^{-2}\,\mathrm{kg}^{-1} \times 2.0 \cdot 10^{30}\,\mathrm{kg}}{7.0 \cdot 10^8\,\mathrm{m} \times 3.0 \cdot 10^8\,\mathrm{m}\,\mathrm{s}^{-1} \times 3.0 \cdot 10^8\,\mathrm{m}\,\mathrm{s}^{-1}} \sim 2.1 \cdot 10^{-6}.$$

This angle, though small, is possible to detect: The required lens diameter is roughly

$$d \sim \lambda/\theta \sim \frac{0.5 \cdot 10^{-6}\,\mathrm{m}}{2.1 \cdot 10^{-6}} \sim 20\,\mathrm{cm}.$$

The eclipse expedition of 1919, led by Arthur Eddington of Cambridge, tried to measure exactly this effect.

*2.2 Approximate Newtonian analysis*

Now that I've got comfortable with the quantities and numerical magnitudes, I want to figure out the bending angle using Newtonian gravitation and a simple assumption: A light 'particle' is just like a rock, except that it moves damn fast (with speed $c$).

While the speeding rock is near the sun, gravity drags it towards the sun, giving it a downwards velocity and thereby bending the path from the horizontal. The point of nearest approach is at a distance $r$, when gravity acts directly downwards (at other spots, some but not all of the acceleration is downwards). But gravity acts on the rock always, not just when the 'rock' is nearest the sun. Its effect is weaker than at the nearest approach. Its effect is even weaker because the downwards fraction of the acceleration gets smaller as the rock gets farther away from the sun (Figure 3)

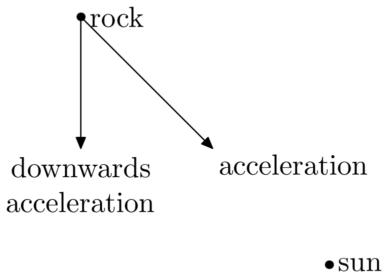


**Figure 3.** *The gravitational acceleration is always directed towards the centre of the sun; the magnitude of the acceleration varies depending on how far the light is from the sun: The acceleration is, from Newton's theory of gravity, $Gm/r^2$. I'm only interested in the downwards component – that piece is what deflects the light. This component is some fraction of the full acceleration. When the light is far away, the fraction is tiny; the downwards acceleration is the entire acceleration only at the point of closest approach (when the light is right 'above' the sun).*

Since the downwards acceleration changes along the path, the analysis requires calculus. For now I avoid it by using an estimation method: I pretend that the sun affects the 'rock' for a distance $r$ along the path, and the downwards acceleration is the same as at the closest approach. I could have chosen $4r$ or $r/7$ as the magic distance. But $r$ is as good a guess as any, and it is simple. Another estimation principle: Forget the constants! The strength of the sun's pull – its gravitational acceleration – is $a = Gm/r^2$ and it acts for a time $t = r/c$. So the downwards velocity of the light is:

$$v = at = \frac{Gm}{rc}.$$

The bending angle is, for small angles, $v/c$ (Figure 4), or

$$\theta \sim \frac{Gm}{rc^2}, \tag{5}$$

as the dimensional analysis suggested in (3) (but did not prove). This analysis is slothful about the constants, but it confirms the functional form from before.

*2.3 More exact Newtonian analysis*

The last analysis assumed the acceleration was zero until the light got into the 'zone of influence' of the sun, then stayed at the maximum value until it left the zone. In fact, at position $x$,

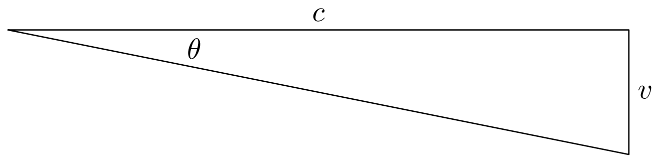


**Figure 4.** *Gravitational acceleration, acting in the magic zone, gives the light a downwards velocity $v$. The sideways velocity of the light is still $c$. As the triangle shows, the bending angle becomes $v/c$. This result assumes that the angle is small, which it will be: The dimensional analysis suggests that the angles will be miniscule!*

the acceleration (towards the sun's centre) is $Gm/(r^2+x^2)$, and the downwards fraction is $r/\sqrt{r^2+x^2}$ (Figure 5).

So

$$a(x) = \frac{Gmr}{(r^2+x^2)^{3/2}}.$$

In a short interval $dx$ the downwards velocity added is

$$dv = a(x) \times \text{time to cross } dx = a(x)\,dx/c.$$

I can now add the contributions from the whole path:

$$v = \int_{-\infty}^{\infty} \frac{Gm}{c} \frac{r}{(r^2+x^2)^{3/2}}\,dx.$$

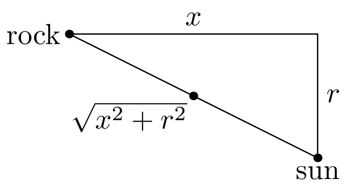


**Figure 5.** *The acceleration towards the centre is $Gm/l^2$, where $l$ is the hypotenuse, $\sqrt{x^2+r^2}$. So $a_{\text{center}} = Gm/(x^2+r^2)$. The downwards fraction is the height of the triangle over the hypotenuse: $r/\sqrt{x^2+r^2}$. So the downwards acceleration is*

$$a = \frac{Gm}{x^2+r^2} \frac{r}{\sqrt{x^2+r^2}} = \frac{Gmr}{(r^2+x^2)^{3/2}}.$$

Rather than keep all the constants around, I'll use a trick based on the previous analysis: Set $G = M = r = c = 1$ for now and put back the correct form at the end. So I evaluate

$$v = \int_{-\infty}^{\infty} (1+x^2)^{-3/2}\,dx.$$

To do this integral, I make the sly substitution $x = \tan\phi$. Then $dx = \mathrm{s}^2\phi\,d\phi$ and the limits become $\pm\arctan\infty$ or $\pm\pi/2$. The rewritten integral is

$$v = \int_{-\pi/2}^{\pi/2} (1+\tan^2\phi)^{-3/2}\ \mathrm{s}^2\phi\,d\phi,$$

or

$$v = \int_{-\pi/2}^{\pi/2} \cos\phi \, d\phi = 2.$$

Now I put back all the missing constants by using the functional form of (5):

$$v = 2\,\frac{Gm}{rc}$$

and the bending angle is

$$\theta = v/c = 2\,\frac{Gm}{rc^2}.$$

### *2.4 General relativity analysis*

Don't worry – I'm not planning to solve the terrible partial differential equations of (1). I'll just quote the result: 4. How can it just be 4? Well, it's 4 just like the Newtonian value is 2, which means that general relativity predicts

$$\theta_{\text{gr}} = 4\,\frac{Gm}{rc^2}. \qquad (6)$$

## 3 History of this test

For many years Einstein believed that his theory of gravity would predict the Newtonian value, which turns out to be 0.87 arcseconds for light just grazing the surface of the sun. The German mathematician, Soldner, derived the same result in 1803. Fortunately for Einstein's reputation, the eclipse expeditions that went to test his (and Soldner's) prediction got rained or clouded out. By the time an expedition got lucky with the weather (Eddington's in 1919), Einstein had invented a new theory of gravity, which predicted 1.75 arcseconds. The goal of Eddington's expedition was to decide between the Newtonian and general relativity values. The measurements are difficult, and the results were not accurate enough to decide which theory was right. But 1919 was the first year after the World War, in which Germany and Britain had fought each other almost to oblivion. A theory invented by a German, confirmed by an Englishman (from Newton's university, no less) – such a picture was comforting after the trauma of war, so the world press and scientific community saw what they wanted to: Einstein vindicated! A proper confirmation of Einstein's prediction came only with the advent of radio astronomy, which could measure small deflections accurately. I leave you with this puzzle: If the accuracy of a telescope is $\lambda/d$, how could radio telescopes be more accurate than optical ones, since radio waves have a longer wavelength than light has?!