OSCILLATIONS AND WAVES

V. Gibson (A) and J. R. Batley (B)

Examples Sheet

Harder questions are indicated by an asterisk (*).

Simple Harmonic Motion

1. (a) A mass m is supported from a light spring with spring constant k. Show that the mass can undergo vertical oscillations of period $T = 2\pi\sqrt{m/k}$.

   (b) Explain briefly why the gravitational acceleration g does not enter into the expression for T.

   (c) What will the period become if the length of the spring is halved ?

   (d) What will the period become if the two half-springs are used in parallel ?

2. A mass M is supported by a smooth table and connected to two light horizontal springs as shown in the diagram below. Each spring is of natural length L and has a spring constant k.

3. (a) An object of mass m is pivoted about a horizontal axis a height h above its centre of mass, and undergoes small amplitude oscillations in a vertical plane. Show that the angular frequency ω of these oscillations is given by $\omega^2 = mgh/(I + mh^2)$ where I is the moment of inertia of the object about a parallel axis through its centre of mass.

   (b) Find the angular frequency ω of small amplitude oscillations of a thin uniform rod of mass m and length l supported at its upper end.

   (c) The rod in (b) is pivoted instead about a horizontal axis placed a distance h from the centre of the rod. Show that the frequency of vertical oscillations about this axis is a maximum for the choice $h = l/\sqrt{12}$ and find the maximum angular frequency.

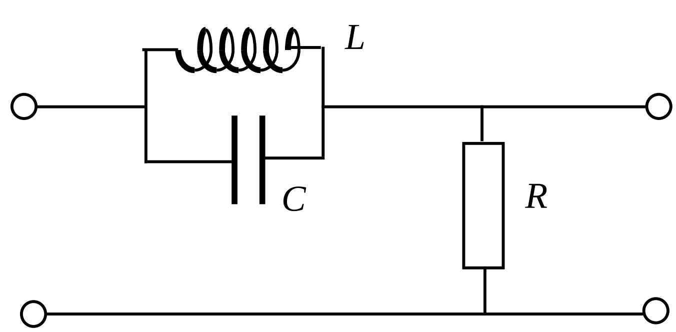
4. A bungee jumper (considered as a point of mass M) jumps from a river bridge of height H, just touches the surface of the water below the bridge, and finally comes to rest a distance $H/4$ above the water's surface. Find the spring constant and unstretched length of the bungee rope.

5. A light bar of length L carries at its ends masses M and m and is suspended horizontally at its centre of mass from a vertical torsion wire of modulus (i.e. the couple needed to give the wire a twist of one radian) τ. The point of suspension is fixed in space. A light horizontal spring of spring constant k is attached to the mass m and has its other end fixed to a point at the same vertical height as the bar. In equilibrium, the bar and the spring are mutually perpendicular. Use the "energy method" to show that the angular frequency of small oscillations is given by

$$\omega^2 = \frac{\tau(M+m)^2 + kL^2M^2}{L^2 mM(M+m)}.$$

Electronic Components

6. The band rejection circuit shown below is used to selectively remove a range of input frequencies.

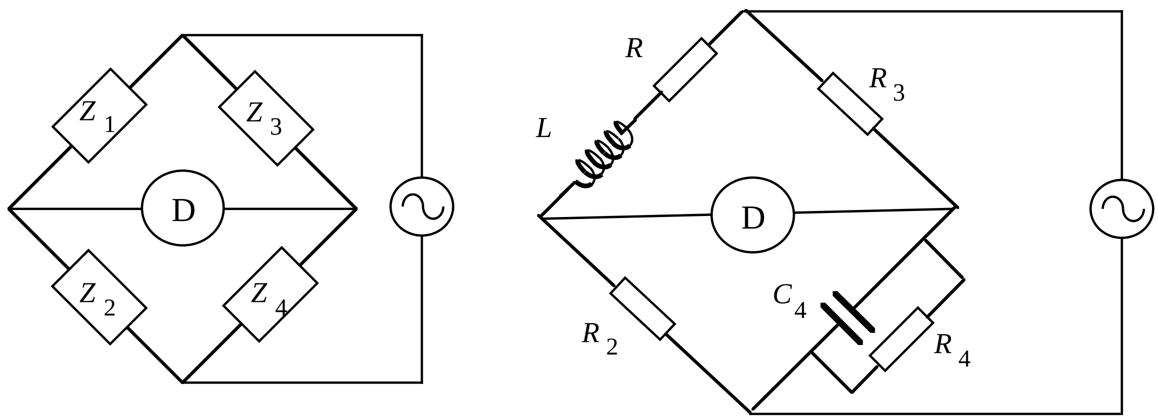


   (a) Obtain an expression for the output voltage V_o when a sinusoidal input voltage V_i of angular frequency ω is applied to the circuit.

   (b) Find the value ω_0 of ω for which the output voltage is zero.

(c) Show that $|V_o| = |V_i|/\sqrt{2}$ when $\omega = \omega_0 \pm (2RC)^{-1}$, provided $(2RC)^{-1} \ll \omega_0$. Sketch $|V_o/V_i|$ as a function of ω.

7. The 'bridge' circuits below are said to be 'balanced' when the detector D registers no voltage difference between its terminals.

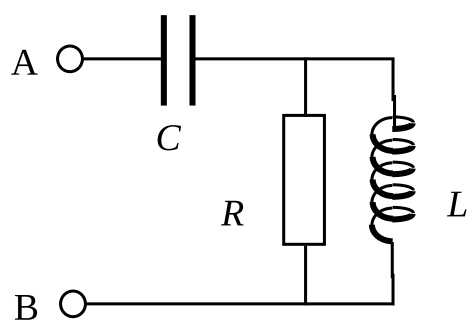


(a) For the left hand circuit, where the components have arbitrary complex impedances, show that the balance condition can be written as $Z_1/Z_2 = Z_3/Z_4$.

(b) Hence for the right hand circuit (known as the Maxwell Bridge) find formulae for R and L in terms of the other components when the circuit is balanced.

(c) A bridge has $R_2 = R_3 = 300\,\Omega$, and unknown R and L. Balance is obtained by adjusting R_4 to $9\,\mathrm{k}\Omega$ and C_4 to $1.0\,\mu$F. Find the values of R and L.

8. (a) Find the complex impedance of the circuit shown below.



(b) If $L = CR^2$ and a sinusoidal voltage of angular frequency $1/\sqrt{LC}$ is applied across AB, show that the current flowing through the circuit is $\pi/4$ out of phase with the applied voltage. Which leads ?

Damped SHM

9. (a) Verify by direct substitution that $x = (A + Bt)e^{-\gamma t}$ is a solution of

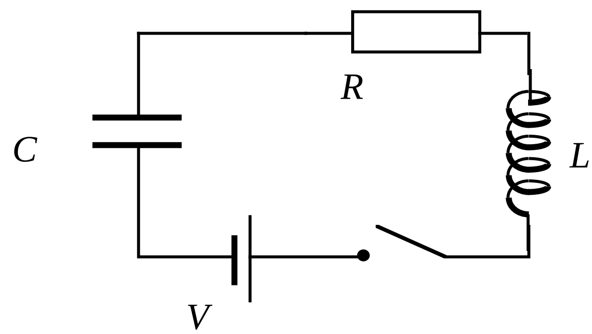
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

for arbitrary constants A and B when $\gamma = \omega_0$, so that the system is critically damped.

(b) In the series LCR circuit shown below, the resistance R is chosen to give critical damping. At time $t = 0$, with the capacitor uncharged and no current flowing, the switch is closed. Show that, at later times, the current is given by

$$I(t) = \frac{V}{L} t e^{-(R/2L)t}$$

and that it reaches a maximum of $2V/(eR)$, where e is the exponential constant. Calculate the total energy dissipated in the resistor, and comment on your result.



10. A mass m is suspended from a light spring of spring constant k, and its displacement x from equilibrium satisfies

$$m\ddot{x} + b\dot{x} + kx = 0,$$

where b is the damping coefficient of the motion.

At time $t = 0$, with the mass stationary and in its equilibrium position, the upper end of the spring is suddenly moved upwards by a distance X. For the case $k = 4b^2/25m$:

(a) Sketch the subsequent motion.

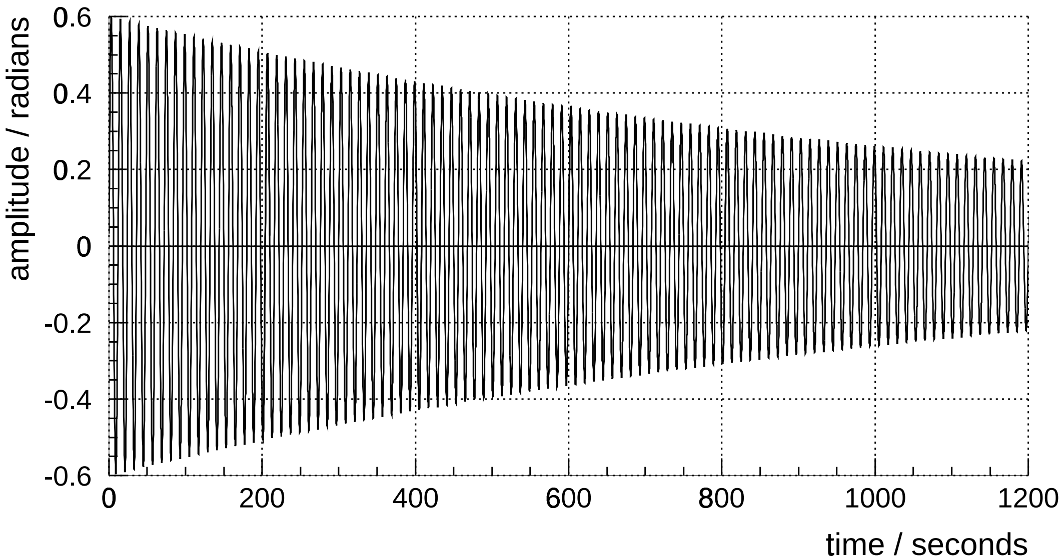
(b) Show that, for large times t, the motion can be approximated by the single exponential term $\frac{4}{3}Xe^{-bt/(5m)}$.

(c) Show that the time required for the mass to move (finally) to within $X/100$ of its new equilibrium position is approximately $24.5m/b$.

Driven SHM

11. Discuss the quality factor Q in oscillating systems and the various ways in which Q can be measured.

12. An antique clock keeps time by means of a torsional oscillator. The torsional bob has moment of inertia $I = 5.9 \times 10^{-5}\,\mathrm{kg\,m^2}$ and is suspended from a thin wire which provides a restoring torque when the bob is rotated away from its equilibrium position. A free (undriven) oscillation of the rotation of the bob *vs* time is shown in the graph below.



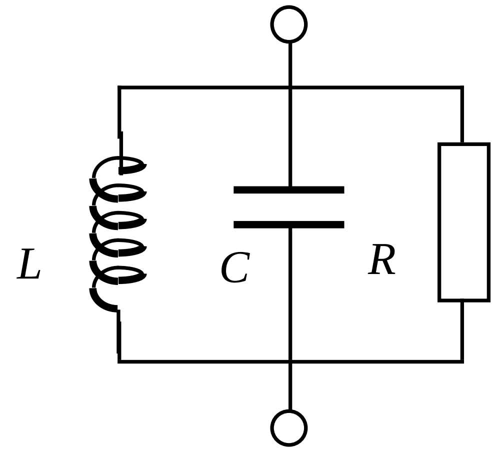
(a) Estimate the angular frequency ω of the oscillation and the restoring torque per unit angular displacement (torsional spring constant) of the wire.

(b) Estimate the quality factor of the oscillation, and the accuracy you would expect of this clock when its oscillation is maintained (driven) by a winding mechanism.

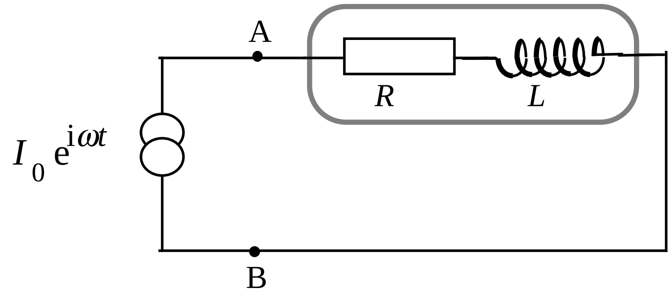
*(c) Explain whether you would expect the clock to run fast or slow when it is placed at the centre of a light turntable which is free to rotate.

13. Which is the best way to push a swing ? Give a reasoned answer in terms of the phase of the applied force.

14. A radio receiver contains the parallel LCR circuit shown below, with $L = 10\,\mu\mathrm{H}$, $C = 2\,\mathrm{nF}$ and $R = 5\,\mathrm{k}\Omega$. If a radio signal induces an alternating voltage V_0 of amplitude $1\,\mathrm{V}$ across the circuit, find the total current and the currents flowing through L, C and R at resonance. (Give both the magnitude, and the phase relative to V_0.) What is the total impedance of the circuit at resonance and is it a maximum or a minimum?



15. The diagram shows a constant-current supply I_0 of angular frequency ω, connected to a coil of inductance L and resistance R. The coil produces a magnetic field into which material samples are placed for investigation.



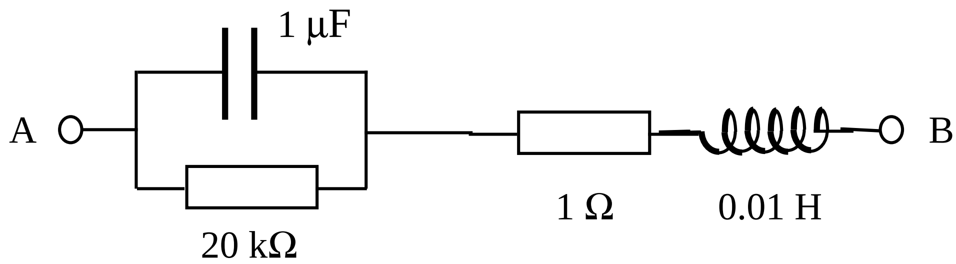
The current through the coil (and hence the magnetic field) can be increased by connecting a suitable capacitor across AB.

(a) If the resistance R is small, show that the current can be increased by a factor of approximately $\omega L/R$. (It may help to consider first the case $R = 0$ to suggest an appropriate value of C.)

(b) Draw a phasor diagram of the currents flowing in the various parts of the circuit.

*(c) Show that, for a general value of R, the current through the coil is a maximum for the choice $C = L/(\omega^2 L^2 + R^2)$, and that the current is then increased by a factor $\sqrt{\omega^2 L^2 + R^2}/R$.

16. The following is equivalent to a series LC circuit in which the capacitor is 'leaky' (represented by a resistance R_1 in parallel with C) and the inductor has a small resistance (represented by a series resistance R_2).



(a) Consider free oscillations in the circuit when the capacitor is initially charged and terminals A and B are connected, and assume that these oscillations are very lightly damped:

(i) Estimate the resonant frequency.

(ii) Divide the average rate of energy dissipation due to capacitor leakage by the maximum energy stored in the capacitor.

(iii) Divide the average rate of energy dissipation in the inductor by the maximum energy stored in the inductor.

(iv) Hence estimate the fraction of the total stored energy dissipated per cycle, and the Quality Factor Q.

(b) Sketch the current response as a function of frequency when an a.c. voltage is applied across AB, and indicate the half-power bandwidth.

*(c) Confirm your answer to part (b) by a direct analysis of the complex admittance $Y \equiv 1/Z$ of this circuit, making suitable approximations to extract the key features.

Feedback

17. The figure below shows the open loop gain of a typical 741C op-amp, which is used as the basis of a non-inverting amplifier circuit, as a function of the frequency f.

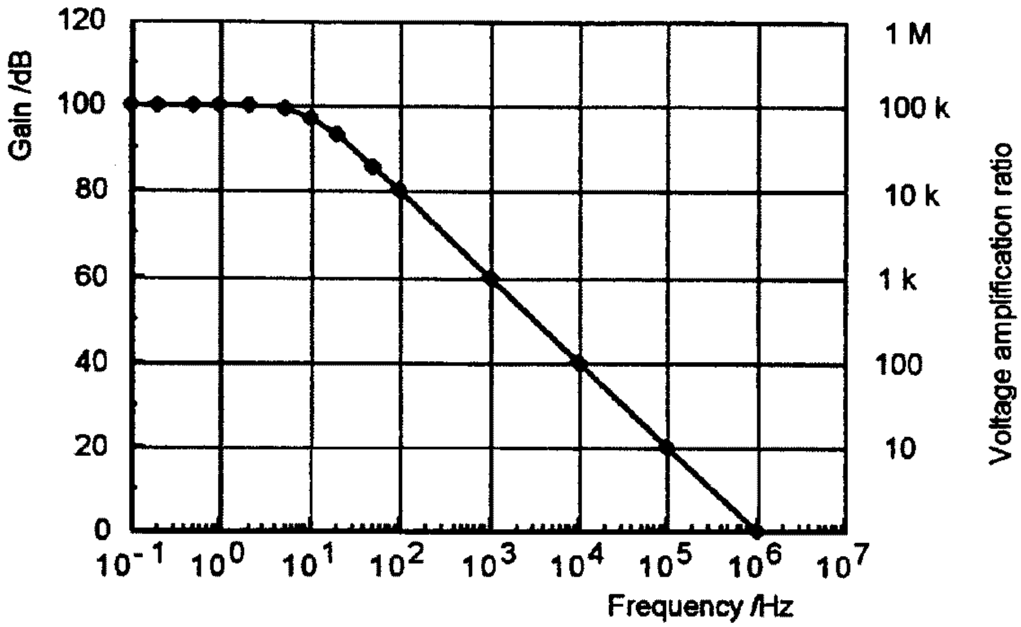
(a) Superimpose on the op-amp gain plot the expected variation of the closed loop gain G of the amplifier circuit with frequency for $\beta = -1/100$, where β is the fraction of the output voltage returned to the input (as negative feedback).

(b) Consider what would happen for other values of β.

(c) Deduce that the product

$$\text{Gain achieved } (G) \times \text{Frequency up to which it is achieved } (\Delta f)$$

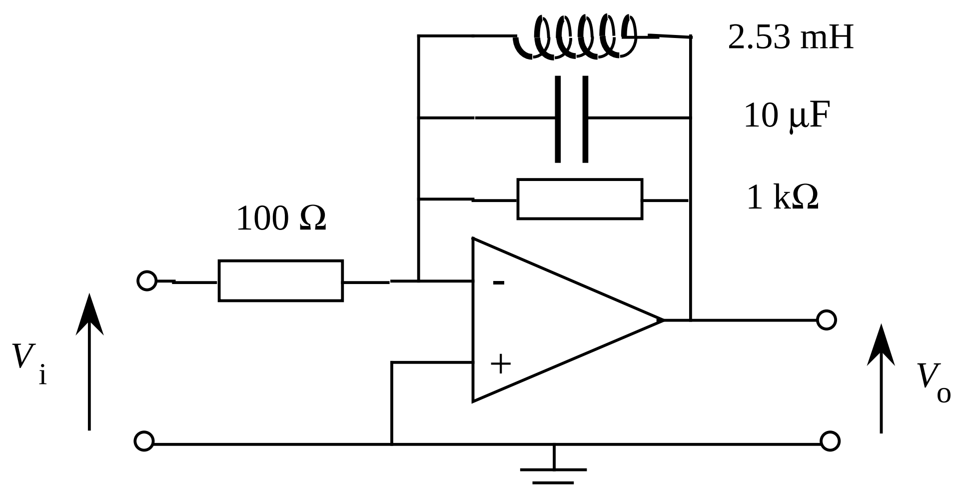
is independent of β, and hence of G (i.e. the product $G \cdot \Delta f$ is a characteristic of the op-amp).



18. (a) For the circuit below, show that

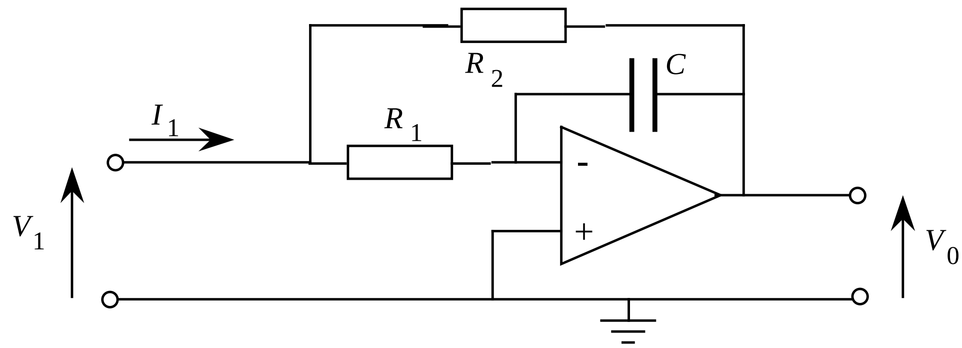
$$\frac{V_o}{V_i} = -\frac{R_2}{R_1} \bigg/ \left[1 + iR_2\left(\omega C - \frac{1}{\omega L}\right)\right]$$

where R_1 is the resistance of the input resistor, R_2, L and C are connected in parallel across the op-amp, and ω is the angular frequency of the input signal.



(b) Hence sketch the amplitude and phase response of the circuit for frequencies between 0 and 100 kHz.

19. By finding the relationship between V_1 and I_1, show that the circuit below behaves effectively as an inductor and a resistor in parallel.



(You may find it helpful to first find the relationship between V_0 and V_1, which does not involve R_2.)

Wave Motion

20. (a) Show that the following mathematical expressions are each different representations of the same travelling wave:

$$\cos\left[2\pi\left(\frac{x}{\lambda} - ft\right)\right], \qquad -\sin\left[\frac{2\pi}{\lambda}(ct - x) - \frac{\pi}{2}\right],$$

$$\mathrm{Re}\left[e^{i(\omega t - kx)}\right], \qquad \mathrm{Im}\left[e^{i(kx - \omega t + \pi/2)}\right], \qquad \mathrm{Re}\left[\frac{1 - i\sqrt{3}}{2} e^{i(\omega t - kx + \pi/3)}\right].$$

($\lambda = 2\pi/k$, $\omega = 2\pi f$, and c is the wave speed.)

(b) Write down an expression describing a travelling wave of the same frequency and wavelength as in (a), but travelling in the opposite direction.

(c) Show that the displacement resulting from the superposition of the travelling waves in (a) and (b) can be expressed as the product of separate functions of x and t alone, and describe the resulting wave motion.

21. The particle displacement of a wave at position x and time t is given by $\mathrm{Re}\left[Ae^{i(\omega t - kx)}\right]$ where $A = 0.1\,\mathrm{m}$, $\omega = 20\pi\,\mathrm{s}^{-1}$ and the wave velocity is $c = 1.0\,\mathrm{m\,s}^{-1}$.

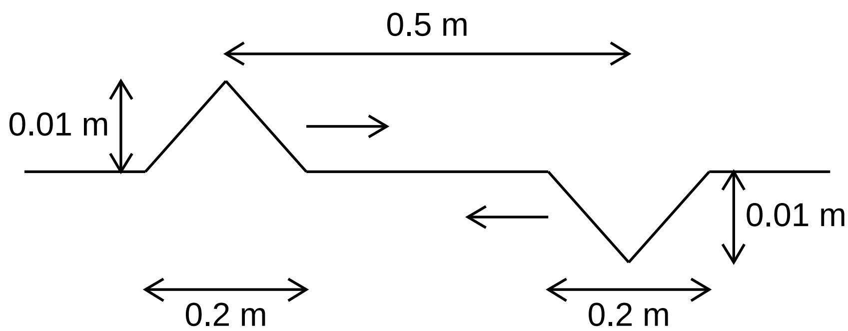
(a) Sketch (on a single diagram) the particle displacement as a function of x at times $t = 0$ and $t = 0.01\,\mathrm{s}$.

(b) Sketch (on a single diagram) the particle displacement as a function of time for positions $x = 0$ and $x = 0.01\,\mathrm{m}$.

(c) Write down an expression for the particle transverse velocity as a function of x and t.

(d) Sketch (on a single diagram) the particle transverse velocity as a function of x at times $t = 0$ and $t = 0.01\,\mathrm{s}$.

22. A string has mass per unit length $\rho = 0.1\,\mathrm{kg\,m^{-1}}$ and is placed under a tension $T = 10\,\mathrm{N}$. At $t = 0$, two pulses travelling in opposite directions are as shown below:



Given that transverse waves on a string travel with velocity $c = \sqrt{T/\rho}$:

(a) Sketch the displacement of the string at times $t = 0$, 0.02, 0.025 and 0.05 s.

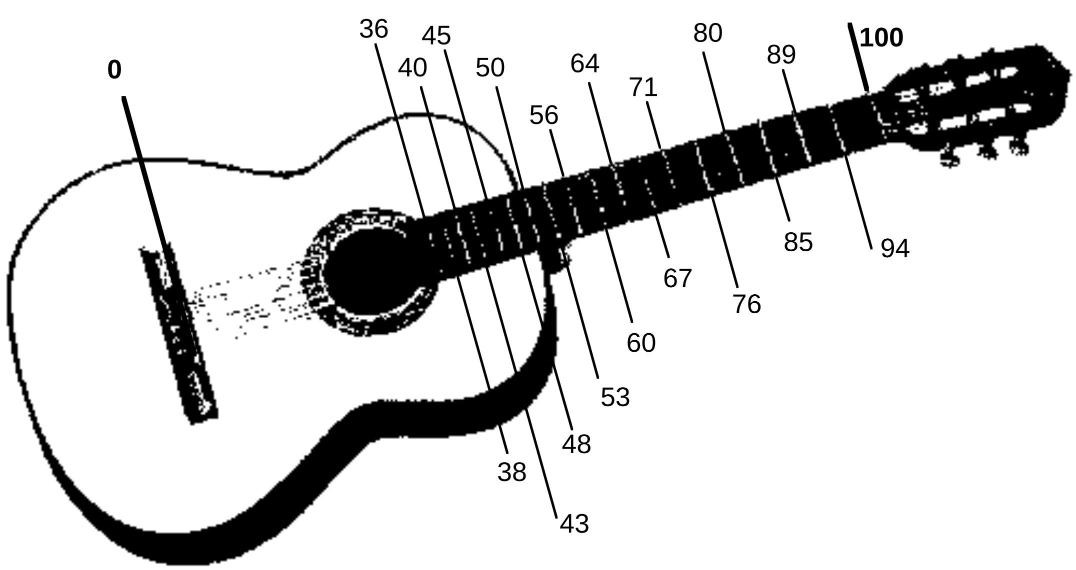
(b) Sketch the transverse velocity of particles of the string at each of these times.

(c) Explain briefly why, at $t = 0.025\,\mathrm{s}$, the energy is entirely kinetic, and calculate the total energy associated with the wave motion.

(d) Hence calculate the potential energy at $t = 0$.

Standing Waves

23. (a) The diagram below shows rough measured values of the fret positions on an acoustic guitar, as a percentage of the distance between the end frets. Can you explain their positions ? Even label the frets for the 'G' string ?



(One octave is a factor of two in pitch; collaboration with a musician is encouraged!)

(b) Guitarists usually presss strings firmly down over a fret with their fingers. However it is possible to obtain different notes by just touching a string above certain frets (and serious violinists exploit this). Can you explain why touching the string above the fret at ~67% gives one octave higher note than does holding the string firmly against this fret? What are the corresponding ratios of pitch expected for the two fingerings for each of the frets at: ~80%, ~76%, ~40% ?

24. The transverse displacement $\psi(x, y, t)$ of a vibrating membrane satisfies the 2-D wave equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}.$$

where c is the wave velocity.

(a) Show by direct substitution that the displacement

$$\psi(x, y, t) = \sin k_x x \sin k_y y \cos \omega t$$

satisfies the 2-D wave equation and derive the connection between ω, k_x, k_y and c.

(b) A drumskin is rigidly held on the edges of a square of side a. Deduce the allowed values of k_x and k_y, and corresponding frequencies $f = \omega / 2\pi$, for transverse vibrations of the drumskin.

(c) List the four lowest frequencies of vibration of the drumskin and, for each frequency, sketch all the corresponding vibration patterns. (Indicate the positions of nodes and anti-nodes, and the relative sign of the displacement over the area of the drumskin.)

1. (c) $T/\sqrt{2}$; (d) $T/2$.
2. (a) $\omega^2 = 2k/M$; (b) $\omega^2 = k/M$.
3. (b) $\omega^2 = 3g/2l$; (c) $\omega^2 = \sqrt{3}g/l$.
4. $23.3Mg/H$, $0.707H$.
6. (b) $\omega_0 = 1/\sqrt{LC}$.
7. (c) $10\,\Omega$, $90\,\mathrm{mH}$.
9. (b) $CV^2/2$.
12. (a) $0.52\,\mathrm{rad\,s^{-1}}$, $1.6\times10^{-5}\,\mathrm{kg\,m^2\,s^{-2}\,rad^{-1}}$; (b) 314, 0.32%.
14. $|I| = |I_R| = 0.2\,\mathrm{mA}$, $|I_L| = |I_C| = 14\,\mathrm{mA}$, $Z = 5\,\mathrm{k\Omega}$.
16. (i) $10^4\,\mathrm{rad\,s^{-1}}$; (ii) $1/R_1C$; (iii) R_2/L; (iv) 66.7; (b) $150\,\mathrm{rad\,s^{-1}}$.
19. $V_0/V_1 = -(i\omega CR_1)^{-1}$, $L = CR_1R_2$, $R = R_1R_2/(R_1+R_2)$.
22. (c) $0.04\,\mathrm{J}$; (d) $0.02\,\mathrm{J}$.
23. (b) 4, 3, 2
24. (c) $c/2a \times \left(\sqrt{2}, \sqrt{5}, \sqrt{8}, \sqrt{10}\right)$.