Problem set 8

Pairs on Wednesday please hand in to my pigeonhole by Wednesday 10am, Friday pairs by Thursday 1pm. Clearly explain your reasoning. Worry not about the length of this set; there are four problems with much guidance (hence the length).

1 *Singing logarithms*
Read the attached sheet on approximating logarithms, and use the method to compute 3^8 and $\log_{10} 5$. How accurate are the values? Make up four more computations in which logarithms would aid the computation; use the method to do the computations.

2 *Adiabatic or isothermal sound waves?*
Newton was the first to work out the speed of sound. He found that $c_s = \sqrt{P/\rho}$. Today we would deduce the speed by deriving and solving the wave equation, which is a partial differential equation for the pressure $p(x,t)$. When Newton derived the speed, regular derivatives were barely understood and partial derivative were unimagined.

Newton's formula implicitly assumes the compressions and rarefactions that constitute a sound wave are isothermal. (An adiabatic compression happens too quickly for heat to flow and thereby to equalise the temperature with the neighbouring rarefaction.) Are the compressions or rarefactions isothermal or adiabatic?

a) To decide, consider a sound wave with angular frequency ω, which is $f/2\pi$. (Angular frequency usually makes for more accurate estimates than regular frequency does.) Roughly how long does a compression last? Call this time t_c. The size of the compression region is roughly c_s/ω, which is usually called $ƛ$. Roughly how long does it take the heat in this region to diffuse outside this region? Call this time t_d. Hint: In a gas, the molecular-diffusion constant D is roughly equal to the heat-diffusion constant κ. Sketch t_c and t_d as functions of ω (on the same graph). What is special about the intersection frequency (call it ω_0)? What is ω_0 as a function of κ and c_s?

b) From Set 6, Question 4d, you know that $\kappa \sim lc_s$, where l is the mean free path. Actually, you found that $D \sim lc$, where c is a typical molecular speed, but $c \sim c_s$, and $D \sim \kappa$. If τ is the mean free time, then $l \sim c_s\tau$. Use this relation to simplify your expression for ω_0.

c) Is a sound wave of frequency $f = 256\,\text{Hz}$ adiabatic (this tone is roughly middle C)? Therefore decide whether Newton's implicit assumption is correct.

d) Now decide experimentally. Compute c_s for air at sea level using Newton's formula. The adiabatic speed is given by

$$c_s^{\text{adiabatic}} = \sqrt{\gamma}\, c_s^{\text{isothermal}},$$

where γ is the ratio of specific heats c_p/c_v, which is roughly 1.4 for dry air. How closely do the two speeds match the actual speed of sound?

3 *Teacup spindown*
You stir your afternoon tea to mix the milk (and sugar if you have a sweet tooth). Once you remove the stirring spoon, the rotation starts to slow. What is the spindown time τ? In other words, how long before the angular velocity of the tea has fallen by a significant fraction?

To estimate τ, consider a physicist's idea of a teacup: a cylinder with height L and diameter L, filled with liquid. Why does the rotation slow? Tea near the edge of the teacup – and near the base, but for simplicity neglect the effect of the base – is slowed by the presence of the edge (the noslip boundary condition); the edge produces a velocity gradient. Because of the tea's viscosity, the velocity gradient produces a force on any piece of the edge; this force tries to spin the piece in the direction of the tea's motion. The piece exerts a force on the tea, which is equal in magnitude and opposite in sense: The edge slows the rotation.

a) In terms of the total viscous force F and of the initial angular velocity ω, estimate the spindown time. Hint: Consider torque and angular momentum. (Feel free to drop any constants, such as π and 2, by invoking the Estimation Theorem: $1 = 2$.)

b) You can estimate F with the idea that

$$\text{viscous force} \sim \rho\nu \times \text{velocity gradient} \times \text{surface area}. \tag{3.1}$$

Here $\rho\nu$ is η. The more familiar viscosity is η, known as the dynamic viscosity. The more convenient viscosity is ν, the kinematic viscosity. (To see why ν might be more convenient than η, work out the dimensions of ν.) The velocity gradient is determined by the size of the region in which the the edge has a significant effect on the flow; this region is called the boundary layer. Let δ be its thickness. Estimate the velocity gradient near the edge, and use (3.1) to estimate F.

c) Put your expression for F into your earlier estimate for τ, which should now contain only one quantity that you have not yet estimated (the boundary-layer thickness).

d) You can estimate δ using your knowledge of random walks. The boundary layer is a result of momentum diffusion; just as D is the molecular-diffusion coefficient, ν is the momentum-diffusion coefficient. In a time t, how far can momentum diffuse? This distance is δ. What is a natural estimate for t? (Hint: After rotating 1 radian, the fluid is moving in a significantly different direction than before, so the momentum fluxes no longer add.) Therefore estimate δ.

e) Now put it all together: What is τ?

f) Stir some tea and estimate τ_{exp}. Compare this value with the value predicted by your theory. In water (and tea is roughly water), $\nu \sim 10^{-6}\,\text{m}^2\,\text{sec}^{-1}$.

4 *Stokes' law*

You can use ideas from the previous problem to derive Stokes' formula for drag at low speeds (more precisely, at low Reynolds' number). Many weeks ago, we derived the result from dimensional analysis; here you will find a physical argument.

Consider a sphere of radius R moving with velocity v. Equivalently, in the reference frame of the sphere, the sphere is fixed and the fluid moves past it with velocity v. Next to the sphere, the fluid is stationary. Over a region of thickness δ (the boundary layer), the fluid velocity rises from zero to the full flow speed v. Assume that $\delta \sim R$ (the most natural assumption) and estimate the viscous drag force. Compare the force with Stokes' formula (remember that $\rho\nu = \eta$).