A1 Useful lengths

I don’t expect you to find ways to estimate all of these quantities. Just take a guess, look them up, and make friends with them; see how they connect to other quantities. Figure out a few that grab your interest and that you want to discuss in supervision. For your enjoyment I give methods for most of them.

**Light year**

Using the useful approximation that 1 yr \(\sim \pi \times 10^7\) s,

\[
1 \text{ ly} = c \times 1 \text{ yr} \sim 3 \times 10^8 \text{ m s}^{-1} \times \pi \times 10^7 \text{ s} \sim 10^{16} \text{ m}.
\]

**A4 sheet (length)**

My foot, at least with a shoe on, is almost the same length as a sheet of A4, and my foot is, surprise, about one foot long. A foot is one-third of a yard, or roughly 0.3 m. An alternative method for Americans, who instead of A4 use ‘eight and a half by eleven’ paper (in inches of course), so a sheet is again about as long as one foot (12 inches). You can also figure it out from the A system. All sizes have length/width = \(1/\sqrt{2}\) – what’s the use of that? A sheet of A0 has area 1 m\(^2\), a sheet of A1 has area 1/2 m\(^2\), etc. So sheet of A4 has area 1/16 m\(^2\), or if it were square, 1/4 m \(\times\) 1/4 m. It’s unsquare by a factor of \(\sqrt{2}\) so it is a bit longer than 1/4 m.

**Bacterium**

A bacterium is probably the size of a rod or cone cell in the retina. Light arrives through the pupil and spreads out due to diffraction over an angle \(\theta \sim \lambda/d\), where \(\lambda\) is the wavelength of light and \(d\) is the diameter of the pupil. There is not much point making pixels more closely spaced than the resolution of the lens, and the cones in the eye fit this pattern. So their size is roughly \(\theta l\), where \(l\) is the length of the eyeball, maybe 3 cm. The wavelength of light is roughly 0.5 \(\mu\)m (a useful length to know), and a pupil may have a diameter around 1 cm or a bit less (look at someone nearby to find out). So

\[
\text{cone size} \sim \frac{\lambda}{d} l \sim \frac{0.5 \times 10^{-6} \text{ m}}{10^{-2} \text{ m}} \times 3 \times 10^{-2} \text{ m} \sim 1.5 \times 10^{-6} \text{ m}.
\]

So cells are roughly 1 \(\mu\)m.

**Car**

One can almost lie down in the back seat of a car, so a car is perhaps 2 m wide. Perhaps the car culture is not as widespread in Britain as in America, where many students in high school, through diverse nocturnal activities, learn this approximation – and why it is a bit on the low side. A car looks about twice as long as wide, so 4 m is a reasonable length. A useful experiment is to pace it off, and one pace is typically 1 m.
Human hair (thickness)
Grabbing a handy eyelash, it looks a lot smaller than 1 mm but is easily visible, so maybe 0.3 mm.

A4 sheet (thickness)
A ream of copier paper is about 2 in thick or 5 cm and contains 500 sheets, so $10^{-2}$ cm or $10^{-4}$ m.

Cambridge
It takes me about 30 minutes to cycle across Cambridge, say from the boundary with Girton on Huntingdon Road to past the railway station. I cycle maybe 10 or 15 miles per hour, so Cambridge is about 6 miles long or 10 km.

Distance to the sun (1 AU)
Very hard to estimate just from data you can see around you; so memorize it: $1.5 \times 10^{11}$ m. The ancient Greeks spent a long time trying to work it out (Aristarchus in particular) and estimated that it was about 20 times the distance to the moon. It is more like 400 times.

Radius of the sun
Its diameter subtends about $0.5^\circ$, which is $10^{-2}$ rad, so

$$d \sim 1 \text{ AU} \times 10^{-2} \sim 1.5 \times 10^9 \text{ m}.$$ 

The radius is therefore $7 \times 10^8$ m.

Distance to the moon
It orbits the earth once a month, from which I can find its orbital angular velocity. Its inward acceleration is $a = \omega^2 r$. This acceleration is due to the earth’s gravity, so

$$a = \frac{GM}{r^2},$$

where $M$ is the earth’s mass and $r$ is the distance to the moon. Oh, no, I have to estimate the earth’s mass! But there’s a trick, again using scaling. This $GM/r^2$ is an acceleration. Is it large or small? Never mind that question, it was just a way to remind you of the moral from 28 January’s lecture: Nothing with dimensions is small or large in itself, it needs to be compared to another quantity of the same type. I want another acceleration; the only one that springs to mind is the acceleration due to gravity at the earth’s surface:

$$g = \frac{GM}{R^2},$$

where $R$ is the radius of the earth. Then

$$\omega^2 r = \frac{GM}{r^2} = \frac{GM}{R^2} \frac{R^2}{r^2} = \frac{gR^2}{r^2}.$$
So

\[ r = \left( \frac{gR^2}{\omega^2} \right)^{1/3}. \]

The radius of the earth is on the list a bit later and is \(6 \times 10^6\) m. The angular velocity is

\[ \omega = \frac{2\pi}{1 \text{ month}} \sim \frac{6}{(1/12) \times \pi \times 10^9 \text{ s}} \sim 2.4 \times 10^{-6} \text{ rad s}^{-1}. \]

Putting in all the numbers,

\[ r \sim \left( \frac{10 \text{ m s}^{-2} \times 36 \times 10^{12} \text{ m}^2}{5 \times 10^{-12} \text{ s}^{-2}} \right)^{1/3} \sim 70^{1/3} \times 10^8 \text{ m}. \]

The cube root of 70 is roughly 4, so

\[ r \sim 4 \times 10^8 \text{ m}. \]

**Diameter of hydrogen**

Dimensional analysis to the rescue. I’ll work out the radius \(r\). It depends on quantum mechanics (without QM, the electron would radiate and spiral into the nucleus) so I include \(\hbar\). Did I hear someone say electron? So it should depend on the electron charge \(q\) and that nasty constant \(\epsilon_0\). Because all forces and energies depend on the combination \(q^2/4\pi\epsilon_0\), the electron charge and \(\epsilon_0\) travel together in that form. The electron mass says how effectively electrostatics can accelerate it, so I better put \(m\) in.

The dimensions:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Dimensions</th>
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<tbody>
<tr>
<td>(r)</td>
<td>L</td>
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<tr>
<td>(m)</td>
<td>M</td>
</tr>
<tr>
<td>(\hbar)</td>
<td>ML(^2)T(^{-1})</td>
</tr>
<tr>
<td>(q^2/4\pi\epsilon_0)</td>
<td>ML(^3)T(^{-2})</td>
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The tricky one is the dimensions of \(q^2/4\pi\epsilon_0\), but

\[ \frac{q^2}{4\pi\epsilon_0 r^2} \]

is a force so the dimensions of \(q^2/4\pi\epsilon_0\) are those of force times length squared. Only one combination of \(m\), \(\hbar\), and \(q^2/4\pi\epsilon_0\) has dimensions of length, so

\[ r \sim \frac{\hbar^2}{mq^2/4\pi\epsilon_0}. \]

That result turns out to be exact (the missing constant is 1) as you’ll learn in the quantum mechanics course. You can just put in the constants from a table, or you can use the trick

\[ \hbar c \sim 2000 \text{ eV Å}, \]

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which is good to 1 percent. (I remember it as ‘200 MeV fermis’ and shift powers of ten between the energy and length units). But the expression for $r$ has no $c$ in it, so what use is the value for $\hbar c$? Ah, but I can insert $c^2$ and take it out:

$$r \sim \frac{\hbar^2 c^2}{mc^2 q^2/4\pi\varepsilon_0}.$$  

Look how $mc^2$ shows up – I don’t have to even look up the electron mass as long as I remember its rest energy. A related useful value is the proton rest energy, roughly 1 GeV, and a proton is about 2000 times as massive as an electron (no one knows why but so it is) so an electron rest energy is 0.5 MeV. What about the $q^2/4\pi\varepsilon_0$? Another useful combination is the famous fine-structure constant $\alpha$:

$$\alpha \equiv \frac{q^2/4\pi\varepsilon_0}{\hbar c} \approx \frac{1}{137}.$$  

This constant is fundamental to quantum electrodynamics and is the dimensionless, and therefore only measure of how strong the electric charge is.

So

$$r \sim \frac{\hbar c}{q^2/4\pi\varepsilon_0} = \frac{\hbar c}{mc^2} \times 137 \times \frac{2000 \text{ eV Å}}{500 \text{ keV}} \sim 0.5 \text{ Å}$$

and the diameter is roughly 1 Å.

**Radius of the earth**

America is roughly three time zones wide and is about 3000 miles wide, so 24 time zones would be 24,000 miles. How do I know America is 3000 miles wide? It takes about 5 hours to fly across and planes go roughly the speed of sound, which is about 600 miles per hour. Since $\pi = 3$ the radius is 4,000 miles or $6 \times 10^6$ m. Sorry for sinning by using miles. Feel free to rewrite the solution in kilometers; but the numbers come out so easily at first in miles (1 time zone = 1000 miles).

An alternative method is that a nautical mile is one minute of arc on the globe, so the earth’s circumference is

$$1 \text{ nautical mile} \times \frac{60 \text{ minutes}}{\text{degree}} \times \frac{360 \text{ degrees}}{\text{full circle}} \sim 20000 \text{ mi}.$$  

And nautical miles are nearly the same size as regular miles.

**Football pitch (length)**

Perhaps a UK football pitch is like an American football field, which is standard at 100 yards, so roughly 100 m is reasonable.

**Height of atmosphere**

The atmosphere doesn’t have a well-defined height since it just gets thinner and thinner, but a reasonable definition is how high before it gets ‘significantly’ thinner, say
by a factor of $e$. From the isothermal atmosphere model from last year, which produces the magic Boltzmann factor,
\[ \rho = \rho_0 e^{-mgh/kT}, \]
where $\rho_0$ is the sea-level density and $m$ is the mass of a molecule of air, and $k$ is Boltzmann’s constant. So the height $H$ at which the density has dropped by a factor of $e$ is when $mgH/kT = 1$ or
\[ H = \frac{kT}{mg} = \frac{RT}{m_{\text{molar}} g}, \]
where I multiplied by Avogadro’s number on top and bottom in the last step. Then, with the molar mass of air roughly 28 g (it’s mostly diatomic nitrogen),
\[ H \sim \frac{8 \text{ J mol}^{-1} \text{ K} \times 300 \text{ K}}{2.8 \times 10^{-2} \text{ kg mol}^{-1} \times 10 \text{ m s}^{-2}} \sim 10 \text{ km}. \]

A bogus method – or rather a fine method that works for subtle reasons – is to estimate how high a molecule could go unimpeded if launched with the thermal velocity. From physics you studied long ago, $H \sim v^2/g$, where $v \sim 300 \text{ m s}^{-1}$ is a crude estimate of the thermal velocity (made by using the sound speed). So
\[ H \sim \frac{10^5 \text{ m}^2 \text{ s}^{-2}}{10 \text{ m s}^{-2}} \sim 10 \text{ km}. \]
Hey, not bad. The error in the thermal speed cancelled the missing factor of 2 in $H \sim v^2/g$.

As an alternative method, the pressure at sea level is $p_0 \sim 10^5 \text{ Pa}$ and
\[ p = \rho gh. \]
This formula works only if the density is constant, which it is not in the atmosphere, but if we assume that the atmosphere has constant density up to the height $H$ then stops abruptly:
\[ H \sim \frac{p_0}{\rho g} \sim \frac{10^5 \text{ Pa}}{1.3 \text{ kg m}^{-3} \times 10 \text{ m s}^{-2}} \sim 10 \text{ km}. \]

**Mean free path of air molecules**

The method in lecture used $n\sigma l \sim 1$ where $n$ is number density, $\sigma$ is the cross-section, and $l$ is the mean free path. The number density follows from the molar volume of 22 litres. With $\sigma = \pi d^2$ and $d \sim 3.5 \text{ Å}$ for air, the mean free path is
\[ l \sim \frac{1}{n\sigma} \sim \frac{2.2 \times 10^{-2} \text{ m}^3}{6 \times 10^{23}} \times \frac{1}{3 \times 12 \times 10^{-20} \text{ m}^2} \sim 10^{-7} \text{ m}. \]

**Average depth of oceans**

Hard to say. I’d guess somewhere between 1 and 10 miles or 1 and 10 km. It turns out to be about 4 km. If you find a way to figure it out from more obvious numbers, let me know.
Height of Mt Everest

Planes have trouble flying over the Himalayas, and planes fly at 10 km or so (they often display their cruising altitude now on those computer monitors), so roughly 10 km for the height of Everest seems reasonable.

A2 Age of sun by a different mechanism

Chemical energy causes very little change in mass (use \( E = mc^2 \) to check) so the sun won’t disappear just because its alleged chemical energy has been mostly used up. Chemical energies are roughly 5 or 10 kcal g\(^{-1}\) or 20–40 MJ kg\(^{-1}\) (check a packet of crisps) with petrol toward the higher end. So the energy available is

\[
E \sim 4 \times 10^7 \text{ J kg}^{-1} \times 2 \times 10^{30} \text{ kg} \sim 10^{38} \text{ J}.
\]

The power output of the sun (its luminosity) is on the last solution sheet. Here it is again. The solar flux at the earth’s orbit is \( F = 1.3 \times 10^3 \text{ W m}^{-2} \), so integrated over a spherical shell at the earth’s orbit,

\[
P = 4\pi R_{\text{orbit}}^2 F \sim 4 \times 3 \times (1.5 \times 10^{11} \text{ m})^2 \times 1.3 \times 10^3 \text{ W m}^{-2} \sim 4 \times 10^{26} \text{ W}.
\]

This the time the sun could burn is

\[
\tau \sim \frac{E}{P} \sim \frac{10^{38} \text{ J}}{4 \times 10^{26} \text{ W}} \sim 2 \times 10^{11} \text{ s} \sim 10^4 \text{ yr}.
\]

Not very long! That’s when the first cities were founded, so this theory for the sun’s power is really bogus. It was a great mystery for a long, long time how the sun got its power.

A3 Air everywhere

From lecture:

\[
\kappa \sim lv_{\text{thermal}},
\]

where \( l \) is the mean free path and \( v_{\text{thermal}} \) is the thermal speed of the molecules. The thermal velocity is from the thermal energy, which is \( \propto T \), and velocity is the square root of energy so

\[
v_{\text{thermal}} \propto T^{1/2}.
\]

The mean free path is trickier. It depends on number density \( n \) and cross section \( \sigma \):

\[
l \sim \frac{1}{n\sigma}.
\]

The cross section depends only on the molecule, not how hot it is, so it remains fixed. But the number density depends on temperature by the ideal-gas law:

\[
P = nkT.
\]

The number density, if the pressure is held constant, is therefore \( \propto T^{-1} \). The mean free path is therefore:

\[
l \propto \frac{1}{n} \propto T,
\]

and

\[
\kappa \propto T \times T^{1/2} = T^{3/2}.
\]
A4  *Air conditioning on the cheap*

As heat from the room air evaporates the water in the sheet, the room air cools (as if the room were sweating). When I wash a double sheet it feels pretty heavy even after having lots of the water spin out of it (which also means it won’t drip much), maybe \( m \approx 1 \text{ kg} \), most of it water. The energy to evaporate it is \( E = mL_{\text{vap}} \). To cool the room by \( \Delta T \) takes energy

\[ E \sim \rho V c_p \Delta T, \]

where \( \rho V \) is the mass of room air and \( c_p \) is its specific heat. The room may be 3 m high, so

\[ V \sim 30 \text{ m}^2 \times 3 \text{ m} \sim 100 \text{ m}^3. \]

The specific heat of air is \( 7R/2 \) or, since one mole of air has a mass of roughly 30 g:

\[ c_p = \frac{7}{2} \times \frac{8 \text{ J mol}^{-1} \text{ K}^{-1}}{3 \times 10^{-2} \text{ kg}} \sim 10^3 \text{ J kg}^{-1} \text{ K}^{-1}. \]

Putting in all the numbers:

\[ \Delta T \sim \frac{mL_{\text{vap}}}{\rho V c_p} \sim \frac{1 \text{ kg} \times 2 \times 10^6 \text{ J kg}^{-1}}{1 \text{ kg m}^{-3} \times 100 \text{ m}^3 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}} \sim 20 \text{ K}. \]

Hey, not bad. It makes the 30 °C room into a reasonable 10 °C, if the cooling is 100% efficient. Some heat comes from the walls, new heat comes in the window (and from the fan motor), so it won’t work quite that well, but the calculation shows that this form of air conditioning is plausible. An evaporative cooler uses this principle. It’s a fancy box that you pour water into and then use a fan to blow dry outside air past it. David MacKay used one to cool his room when he was a postgraduate in Pasadena, California (hot summers).

A5  *Number sense*

\[ \frac{1}{0.98} \]

\[ \frac{1}{0.98} = \frac{1}{1 - 0.02} \approx 1 + 0.02 \]

because \((1 + x)(1 - x) \approx 1\).

\[ 1.1^{10} \]

The binomial theorem, cut off after one term, gives

\[ (1 + x)^{10} \approx 1 + 10x, \]

or 2 in this case. But the other terms are large enough to matter. So instead take the logarithm:

\[ \ln 1.1^{10} = 10 \ln 1.1 = 10 \ln(1 + 0.1) \approx 10 \times 0.1 = 1. \]

So

\[ 1.1^{10} = e \approx 2.718. \]
From the first number-sense problem, $0.9 \approx 1/1.1$ so

$$0.9^{70} \approx 1.1^{-70} = (1.1^{10})^{-7}.$$  

Using the result for $1.1^{10}$ and the useful fact given in the problem, that $2^7 = 10^3 = e^7$:

$$0.9^{70} \approx e^{-7} = 0.001.$$  

$(1 + x)^2 \approx 1 + 2x$, so if $2x = 0.05$ then $x = 0.05/2$ and

$$\sqrt{1.05} \approx 1.025.$$  

In other words, if you increase the area of a square by 5%, you increased its sides by 2.5%. In general, if $f = x^n$, then taking logarithms,

$$\ln f = n \ln x,$$  

and then differentiating:

$$\frac{df}{f} = n \frac{dx}{x}.$$  

In easy-to-remember English:

fractional change in $f = n \times$ fractional change in $x$.  

I take out the big part that I can do easily: $\sqrt{49}$. So

$$\sqrt{50} = 7 \sqrt{1 + \frac{1}{49}} \approx 7 \times (1 + 1/98) = 7 + \frac{1}{14} \approx 7.07.$$  

Or using the easy-to-remember English above: To go from 49 to 50 is an increase of 2%, so to go from $\sqrt{49}$ to $\sqrt{50}$ is an increase of 1%:

$$\sqrt{50} \approx 7 + 1\% = 7.07.$$  

$$\ln 2000 = \ln 1000 + \ln 2 = 7 + \frac{7}{10} = 7.7$$  

since $\ln 1000 \approx 7$ (the useful fact) and $2^{10} = 1000$ implies that $\ln 2 = 0.1 \ln 1000$.  

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\[ \ln 3 \approx e + 0.3 \approx e(1 + 0.1) \text{ so} \]
\[ \ln 3 \approx \ln e + \ln(1 + 0.1) \approx 1.1. \]

\[ \sin 7 \]
I take away the \( 2\pi \approx 6.3 \) first and get
\[ \sin 7 = \sin(7 - 2\pi) \approx \sin 0.7 \approx 0.7 \]
and \( \sin x \approx x \) for even moderately large \( x \). The next correction is \( x^3/6 \), which is tiny even when \( x = 0.7 \).

C1 Light bulb
I have two unknowns, length \( l \) and diameter \( d \). So I need two equations. The first comes from the resistance of the filament, since I know the power and voltage. The second comes from the power radiated by a 3000K source (using the Stefan–Boltzmann law).

The resistivity \( \rho \) is a property of the material and is independent of how much there is. The resistance \( R \) depends on the length and thickness. If I double the length of a wire, I have put two resistors in series and so doubled the resistance. So \( R \propto \rho l \). Furthermore, if I run two wires in parallel, I double the cross-sectional area \( A \) and halve the resistance, so \( R \propto \rho/A \). Putting the two together:
\[ R \propto \rho l/A. \]

Look at the dimensions of \( \rho \): resistance times length. And \( R \) is just resistance. So the left and right sides above have the same dimensions, and
\[ R \sim \rho l/A, \]
where the \( \sim \) means that the two sides have the same dimensions but maybe are off by a (dimensionless) constant. This constant turns out to be 1 because resistivity is defined to make it 1:
\[ R = \rho l/A. \]

The only time such ‘constants by definition’ are not 1 is when a competing equation has a slightly different structure and prevents both constants from being 1 at the same time (the \( 1/3 \) in the equation for thermal diffusivity is an example of this situation).

The resistance itself I do not know, but the light-bulb power \( P = V^2/R \) where \( V \) is the mains voltage, so
\[ P = \frac{V^2}{\rho l/A} = \frac{V^2 A}{\rho l}. \]

Both \( P \) and \( V \) are known quantities, and \( A \sim d^2 \), so I now get one equation for \( l \) and \( d \):
\[ \frac{l}{d^2} = \frac{V^2}{\rho P}. \]
The second equation comes from Stefan–Boltzmann:

\[ P = \text{area} \times \text{flux} \sim \pi ld \times \sigma T^4, \]

where \( \sigma \) is the Stefan–Boltzmann constant. The temperature is known: \( T \sim 3000 \text{ K} \). Remember that the sun, which burns hotter than a lightbulb and therefore gives off bluer light, has \( T_{\text{sun}} \approx 6000 \text{ K} \), so 3000 K is reasonable. The second equation is then

\[ P \sim \pi ld \times \sigma T^4 \]

or

\[ ld \sim \frac{P}{\pi \sigma T^4}. \]

Squaring this equation and multiplying by first one:

\[ l \sim \left( \frac{P}{\pi^2 \sigma^2 T^8} \frac{V^2}{\rho} \right)^{1/3} \]

You should check that the units work! Putting in numbers:

\[ l \sim \left( \frac{6 \times 10^1 \text{ W}}{10^1 \times 3.6 \times 10^{-15} \text{ W}^2 \text{ m}^{-4} \text{ K}^{-8} \times 6.4 \times 10^{27} \text{ K}^4} \times \frac{5 \times 10^4 \text{ V}^2}{10^{-6} \text{ Ohms}^2 \text{ m}^2} \right)^{1/3} \]

I wrote everything with exponents, even \( \pi^2 = 10 = 10^1 \), to make them easy to track for the next step. First count the powers of 10 inside the cube root: 5 on top and 7 on the bottom, so they give \( 10^{-2} \). The other factors are a 6 \times 5 on top and 3.6 \times 6.4 on the bottom, making roughly 1. So

\[ l \sim 10^{-2/3} \text{ m} \sim 0.2 \text{ m}. \]

For fun figure out that last step without a calculator. How reasonable is this value? If I look in a lightbulb the filament is only 1 or 2 cm. But it’s a coil of a coil, so maybe unwound it is 20 cm.

Now I can get \( d \) from the second equation:

\[ d = \frac{P}{\pi l \sigma T^4} \]

so

\[ d \sim \frac{60 \text{ W}}{3 \times 0.2 \text{ m} \times 6 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4} \times 0.8 \times 10^{14} \text{ K}^4} \sim 20 \mu \text{m}. \]

Hmm, that seems very small. We’ll break a lightbulb and look at the filament under a microscope and see whether it’s right or not.
C2 Ice on lakes

I grew up near a lake frozen in winter so I have an idea of the thickness. If you did not, your children may. See ‘Global warming will plunge Britain into new ice age within decades’, (Independent, 25 January 2004, <http://news.independent.co.uk/uk/environment/story.jsp?story=484490>). The article quotes a study warning that global warming may disrupt the Gulf Stream, in which case England will have the weather of Labrador:

Robert Gagosian, the director of Woods Hole, considered one of the world’s leading oceanographic institutes, said: ‘We may be approaching a threshold that would shut down [the Gulf Stream] and cause abrupt climate changes.’

Without waiting that long for direct experience, you can guess the thickness by knowing that people, especially Canadians playing ice hockey, skate on frozen lakes. You can break a thin crust of ice on a puddle with a gentle tap of the finger. Maybe that layer is a few mm thick. So to walk on a frozen lake it better be a lot thicker than that, maybe a few cm. Skates exert a much higher pressure than you do when walking, because skates have such a thin blade (much smaller than your shoe), so the ice on the lake should be a fair amount thicker for skating than for walking, especially to allow a margin of safety, so at least 10–15 cm and it may get even thicker in a long or cold winter.

Now for a calculation. First I explain what is going on in words, then formalize it into an equation. Heat flows out of the warm lake (at 0°C!) into the cold air. The barrier is the ice. As heat leaves, the water just below the ice freezes (so I’ll use heat of fusion later). As the ice thickens, the heat flows more slowly, so the rate of freezing falls too. A differential equation lurks in here!

Fluxes are power per area, but if I reason about fluxes instead of power itself, then I end up thinking about a unit area of ice and get confused. So I imagine a slab of ice of area $A$ and set up the equations knowing that $A$ will cancel (if it does not, it helps catch a mistake in reasoning). Similarly, the power per area is energy per time, and I imagine the energy transferred in a concrete time $\Delta t$. Let $z(t)$ be the thickness of the ice. Then the energy that flows out of an area $A$ in time $\Delta t$ is

$$E = \text{flux} \times \text{area} \times \text{time} = K \frac{\Delta T}{z} A \Delta t,$$

where $K$ is the thermal conductivity of ice and $\Delta T$ is the temperature difference between the air and water. That energy loss freezes a mass of ice $E/L_{\text{fusion}}$, where $L_{\text{fusion}}$ is the heat of fusion (energy per mass of ice). The volume is mass over density and the thickness is volume over area, so the extra thickness is

$$\Delta z \sim \frac{E}{L_{\text{fusion}} \rho A},$$

so

$$\Delta z \sim \frac{K \Delta T A \Delta t}{L_{\text{fusion}} \rho A}.$$

Hey the areas cancel, great, and

$$\Delta z \sim \frac{K \Delta T \Delta t}{\rho L_{\text{fusion}} z}.$$
In the limit that $\Delta t \to 0$, the differential equation appears:

$$\frac{dz}{dt} \sim \frac{1}{z} \frac{K \Delta T}{\rho L_{\text{fusion}}},$$

whose solution is (leaving aside dimensionless constants)

$$z(t) = \sqrt{\frac{K \Delta T}{\rho L_{\text{fusion}}}} t.$$

Now put in numbers and see whether something reasonable comes out. New Hampshire has been much in the British news lately because the American presidential election holds an early hurdle election (a ‘primary’) there, and most of the news reports mention how frozen New Hampshire is, with temperatures of $-20^\circ C$ being typical and $-30^\circ C$ not uncommon. So I’ll use $\Delta T \sim 25^\circ C$. The thermal conductivity we did in lecture: $K \sim 2 \text{ W m}^{-1} \text{ K}^{-1}$. And the heat of fusion is roughly

$$L_{\text{fusion}} \sim \frac{L_{\text{vap}}}{10} \sim 0.2 \text{ MJ kg}^{-1}.$$

A cold winter lasts a few months, say 3 months or one-fourth of a year:

$$t \sim \frac{1 \text{ yr}}{4} \sim \frac{\pi \times 10^7 \text{ s}}{4} \sim 10^7 \text{ s}.$$

Putting in all the numbers:

$$z \sim \left( \frac{2 \text{ W m}^{-1} \text{ K}^{-1} \times 25 \text{ K}}{10^3 \text{ kg m}^{-3} \times 0.2 \times 10^6 \text{ J kg}^{-1} \times 10^7 \text{ s}} \right)^{1/2} \sim 1.5 \text{ m}$$

That’s deep ice!

D1 Random walks

This I’ll leave you to put together from the lecture material (see the notes for lecture 4) and whatever else you find interesting and useful to you.