Entropy demonstration!

Alan Chapman has built a demonstration for us. In the first act: Using a nice block as the cargo, the cart slides down the plane, bounces off the wall, then travels roughly 80 cm up the plane (Figure 1). In the second act: Using a mess of springs and other junk as the payload, the cart slides down the plane and bangs into the wall, and then travels roughly 65 cm up the plane. Why less than in the first act? Because the impact rattles the junk around, and 15 cm worth of potential energy is lost to the shaking. In the third act, I set the junk cart sit at the bottom next to the wall, and I give it a smack. The junk starts shaking, like in the second act. Sadly, the shaking does not combine into useful energy that moves the cart 15 cm up the hill. You are seeing entropy, or disorder. The energy that goes into shaking the junk is disordered. It will not combine into ordered energy that can do mechanical work. In other words, entropy never decreases. You will see this interpretation of entropy in much greater depth in the statistical physics course, and you can read lots about it in P.W. Atkins, The Second Law.

![Figure 1. Cart sliding down plane. The cart's wheels are nearly frictionless, so it loses hardly any energy to friction. The spring in front is a low-loss bumper for the collision. In this case, the cargo is a single block.](image1)

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Adiabatic expansions and compressions

Here was the question from the end of the last lecture. What happens to a molecule’s speed as the wall closes in and it collides with the wall? The wall moves to the right with speed \( v_{\text{wall}} \) and the molecule moves to the left with speed \( v_x \).

What is the molecule’s change in speed after it collides with the wall?

- a. \( v_{\text{wall}} \)
- b. \( v_x v_{\text{wall}} \)
- c. \( v_x / 2 \)
- d. 0

How do you support or argue against these choices?

For a start, imagine an extreme case: The peaceful little molecule is sitting quietly \( (v_x = 0) \) and the big nasty wall smashes into it. Extreme cases are easy to analyse,
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so start with them. The speed afterwards cannot be zero, because then the molecule would have to pass through the wall. So a zero change in speed cannot be right. Thus choices (b), (c), and (d) are out. Choice (b) is doubly out because its dimensions of velocity squared are bogus.

GRE

Many of you may apply to a PhD program in America. You might even go to one if you don’t mind being fingerprinted by the Ministry of Fatherland Security. Almost every PhD program requires the GRE and the TOEFL (Test of English as a Foreign Language). Once in a while they even require British people to take the TOEFL, as a friend who went to Harvard found out. He refused, and they told him to go home to England, so he said he’d see what the newspapers thought of kicking out an English person for not taking the TOEFL. They let him stay.

So normally you’ll just face the GRE. It has a general part and a subject part. The physics subject test is maybe 80 or 100 multiple-choice questions. In many questions, I could eliminate a lot of choices by using dimensions. For example, one question asked for the ground-state energy of a quantum particle in an unpleasant potential (it might have been \( V = x^4 \)). Who has time to solve the Schrödinger equation in a minute or two? However, one answer had dimensions of energy squared, another had dimensions of area; they were easy to eliminate. Others had the right dimensions but looked like some big mess \( h \).

In the classical limit (extreme cases again!), where \( h \to 0 \), the energy would be infinite! So toss out that choice as well. Just using dimensions and a bit of physical reasoning, you can often quickly eliminate most of the choices, sometimes even all but one choice. Then you save time for the questions where you don’t spot a trick right away.

What about choice (a)?

Back to the wall collision. Only choice (a) survived the extreme-cases arguments. How reasonable is it? In the same extreme case of \( v_x = 0 \), this choice says that the final speed is \( v_{\text{wall}} \): The molecule sticks to the wall. I’ll experiment. I put a light plastic water bottle on the desk and move a coffee mug along. After the collision, the water bottle seems to leap off the mug rather than stick to it. So the final speed should be greater than \( v_{\text{wall}} \). But let’s calculate by changing frames.

Changing frames

In the rest frame of the wall, the molecule approaches with speed \( v \), bounces, and leaves with speed \( v \) (Figure 3). Why the same speed? One reason might be conservation of momentum. Momentum is \( mv \) and the speed is preserved by the collision. The confusion is that I’ve used \( v \) for the speed (the magnitude of velocity). However, momentum is a vector so the \( P = mv \) has a sign. Therefore \( mv \) changes in a collision. Another reason could be conservation of energy, because kinetic energy involves \( v^2 \), which means the sign of the velocity does not matter. If the energy of the ball is preserved by the collision, then the energy of the wall must also be preserved. So the collision doesn’t change the speed of the wall. Since the wall is infinitely massive, this
result is reasonable. So conservation energy produces a decent explanation for why the molecule leaves with the same speed at which it approaches the wall.

In the wall’s frame, the gas molecule approaches, and therefore leaves with speed $v_{\text{wall}} + v_x$. In the lab frame, the molecule leaves with speed $v_{\text{wall}} + 2v_x$, so its change in speed is $2v_{\text{wall}}$. The extra boost is consistent with the table-top experiment where the light water bottle leapt away from the mug. But it’s not one of the choices that I offered you!

Am I careless or nasty?

Most of you thought I was nasty rather than careless. Thanks, I think. And you are right. I left out the right answer to show you a general point. We hear endlessly how UK universities are ‘in trouble’. They do not have enough money when compared with their wealthier American cousins. Suppose you are working in government and are asked to decide whether to (a) increase student fees or (b) cut lecturer salaries. If you stick to the assigned choices, you will miss a third choice: cut management salaries and hire no more managers. So, come up with your answers – and even better, your own questions – before looking at what others offer you!

Examples of this speed change

In football the goal keeper often has to kick the ball across the pitch. Her infinitely massive foot hits the ball with speed $v_{\text{wall}}$. The ball is initially at rest, so its final velocity is $2v_{\text{wall}}$. How fast can you move your foot? Maybe you can move swing your leg one-quarter of an arc in 0.1 s (if you are a professional footballer!). The speed is roughly $10 \text{ m s}^{-1}$, so the ball leaves with speed $20 \text{ m s}^{-1}$. The distance travelled is roughly $v^2/g$ (check it by using dimensions) or 40 m. This distance seems reasonable: It is about one-half of a football pitch.

Another example of this kind of collision is used by NASA. Often a spacecraft, say Voyager, needs a boost along the way, but carrying fuel from earth is too difficult. For one, carrying fuel means a heavier payload for the rocket sending the spacecraft into orbit, which greatly increases the cost. Fortunately the solar system has many infinitely massive, moving walls. For example, Jupiter. The spacecraft approaches Jupiter moving opposite to its orbital motion (Figure 4). In the rest frame of Jupiter, the probe approaches, swings around, and then leaves in (almost) the opposite direction. In the frame of the solar system, the probe gets a kick from Jupiter. This is a gravity slingshot. Puzzle: Where does the extra energy for the spacecraft come from?

A third example of the bounce is the problem we began with: molecules colliding with a wall. The principle applies to microscopic systems (molecules colliding with a wall), mesoscopic systems (kicking a football), and macroscopic systems (gravity slingshots in the solar system): a range of perhaps 20 orders of magnitude in length.
Figure 4. Bouncing off Jupiter. In the lab frame, a spacecraft approaches Jupiter with speed \(v\), as shown in (a). Meanwhile Jupiter is doing its usual orbital motion, giving it speed \(v_J\) in the opposite direction. In Jupiter’s frame, shown in (b), the spacecraft approaches with speed \(v + v_J\), so it bounces off with the same speed. Transforming back to the lab frame shows that the spacecraft leaves Jupiter with speed \(v + 2v_J\).

Fractional change in energy

The speed of the gas molecule went from \(v_x\) to \(v_x + 2v_{\text{wall}}\), which is a fractional change of

\[
\frac{\Delta v_x}{v_x} = \frac{2v_{\text{wall}}}{v_x}.
\]

What is the fractional change in energy in terms of the fractional change in velocity? Since kinetic energy is proportional to \(v^2\), a reasonable conclusion is that the fractional changes behave similarly:

\[
\frac{\Delta E}{E} = \left(\frac{\Delta v_x}{v_x}\right)^2.
\]

Let’s check it in a simple example. Imagine a square with side \(x = 1\) and area \(A = 1\). Now increase \(x\) by 10%: a fractional increase of 0.1 (Figure 5). Does the area increase by \(0.1^2\)? The area becomes \(1.1^2 \approx 1.2\), which is an increase of 20%. Here is another example. Increase \(x\) to 1.05. The area becomes \(1.05^2 \approx 1.1\), so a 5% increase in \(x\) turns into a 10% increase in \(A\). The general rule would then be

\[
\frac{\Delta E}{E} = 2 \left(\frac{\Delta v_x}{v_x}\right).
\]

It still has a 2 but the 2 has come in front as a factor rather than living as an exponent. Here’s a rough derivation of that result using logarithmic differentiation. If \(A = x^2\) then \(\log A = 2 \log x\). Now differentiate:

\[
\frac{dA}{A} = 2 \frac{dx}{x}.
\]

And you can easily generalise this result to higher powers.

But, you ask, the energy formula is slightly different from \(A = x^2\) because it has a constant in front. So let’s try \(A = bx^2\) where \(b\) is a constant. Then \(\log A = \log b + 2 \log x\). Since \(b\) is a constant, the differentiation step is the same:

\[
\frac{dA}{A} = 2 \frac{dx}{x}.
\]

Next time we’ll use the result for fractional change in energy to work out the pressure and volume change when you compress a gas.
Figure 5. Increasing the area of a square. Originally there was a unit square (heavy outline). The side length increases by 10% (not drawn to scale), so the area increases, approximately, by the two rectangles. Their area is 0.2.

**Problem solving**

Here is a description of Los Alamos when the news came back that Hiroshima had been bombed:

He entered the room like a prize fighter. As he walked through the hall there were cheers and shouts and applause all round and he acknowledged them in the fighter’s salute – clasping his hands together above his head as he came to the podium.

The prize fighter was Robert Oppenheimer. Here is Hans Bethe describing a later era at Los Alamos:

Grim as the subject was, it was a most exciting enterprise. The ideas we had about triggering an H-bomb turned out to be all wrong but the intellectual experience was unforgettable.

Solving problems is often lots of fun. But which problems are you solving?