Under-estimation of the UK Tidal Resource

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Abstract

A widely-quoted estimate of the practical UK tidal resource is 12 TWh/y [Black and Veatch, 2005]. I believe this is an underestimate, because it is based on an incorrect physical model of the flow of energy in a tidal wave. In a shallow-water-wave model of tide, the true flow of energy is greater than the Black-and-Veatch flow by a factor of d/h, where d is the water depth and h is the tide's vertical amplitude. The tidal resource may therefore have been underestimated by a factor of about 10.

The widely-quoted estimate of the practical UK tidal resource is 12 TWh/y (equivalent to an average production of 1.4 GW, or 0.5 kWh per person per day) [Black and Veatch, 2005].

In a two-page comment on the DTI Energy Review, Salter [2005] suggests that this standard figure may well be an under-estimate (see also Salter and Taylor [2007]). Salter estimates that the dissipation by friction on the sea bed of the Pentland Firth alone is 100 GW (peak). He argues that turbines could be inserted as a sea-bed substitute there and would deliver up to 40 GW (peak).

In this note, I present back-of-envelope models of tidal power that concur with and amplify Salter's view. In sum, the method used by Black and Veatch to estimate the tidal resource, namely estimating the kinetic energy flux across a plane, is flawed because (except in certain special cases) *the power in tidal waves is not equal to the kinetic energy flux across a plane.*

These back-of-envelope models are not new. Essentially identical models are analysed in greater detail by Taylor [1920].

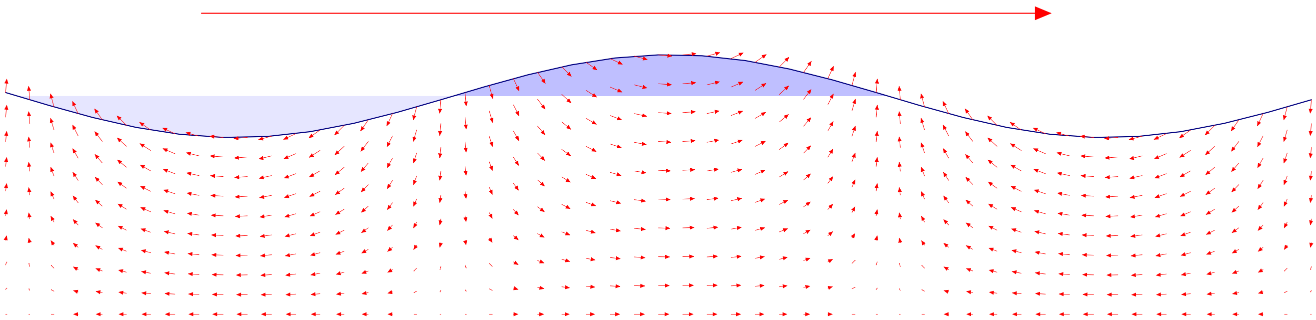

Figure 1. A shallow-water wave. The wave has energy in two forms: potential energy associated with raising water out of the light-shaded troughs into the heavy-shaded crests; and kinetic energy of all the water moving around as indicated by the small arrows. The speed of the wave, travelling from left to right, is indicated by the much bigger arrow at the top. For tidal waves, a typical depth might be 100 m, the crest velocity 30 m/s, the vertical amplitude at the surface 1 or 2 m, and the water velocity amplitude 0.3 or 0.6 m/s.

Tides as tidal waves

Follow a high tide as it rolls in from the Atlantic. The time of high tide becomes progressively later as we move east up the English channel from the Scillys to Portsmouth and on to Dover. Similarly, a high tide moves clockwise round Scotland, rolling down the North Sea from Wick to Berwick and on to Hull. These two high tides converge on the Thames Estuary. By coincidence, the Scottish wave arrives nearly 12 hours later than the one that came via Dover, so it arrives in near-synchrony with the next high tide via Dover, and London receives the normal two high tides per day.

Figure 1 shows a model for a tidal wave travelling across relatively shallow water. This model is intended as a cartoon, for example, of tidal crests moving up the English channel, towards the outer Hebrides, or down the North Sea. The model neglects Coriolis forces. [The Coriolis force causes tidal crests and troughs to tend to drive on the right – for example, going up the English Channel, the high tides are higher and the low tides are lower on the French side of the channel. By neglecting this effect I may have introduced some error into the estimates. The analysis of Taylor [1920] includes the Coriolis effect, and includes the possibility that there is a second tidal wave running in the opposite direction.] The water has depth d. Crests and troughs of water are injected from the left hand side by the 12-hourly ocean tides. The crests and troughs move with velocity

$$v = \sqrt{gd}. \tag{1}$$

We assume that the wavelength is much bigger than the depth. Call the vertical amplitude of the tide h. For the standard assumption of nearly-vorticity-free flow, the horizontal velocity is near-constant with depth. The velocity is proportional to the surface displacement and has amplitude U, which can be found by conservation of mass:

$$U = vh/d. \tag{2}$$

If the depth decreases gradually, the wave velocity v reduces. For the present discussion we'll assume the depth is constant. Energy flows from left to right at some rate. How should this total tidal power be estimated? And what's the maximum power that could be extracted?

One suggestion is to choose a cross-section and estimate the average flux of kinetic energy across that plane. This kinetic-energy-flux method is used by Black and Veatch to estimate the UK resource. In this toy model, we can also compute the total power by other means. We'll see that the kinetic-energy-flux answer is incorrect by a significant factor.

The peak kinetic-energy flux at any section is

$$K_{\mathrm{BV}} = \frac{1}{2}\rho A U^3, \tag{3}$$

where A is the cross-sectional area.

The true total incident power is a standard textbook calculation; one way to get it is to find the total energy present in one wavelength and divide by the period; another option is to imagine replacing a vertical section by an appropriately compliant piston and computing the average work done on the piston. I'll do the calculation both ways. The potential energy of a wave (per wavelength and per unit width of wavefront) is

$$\frac{1}{4}\rho g h^2 \lambda \tag{4}$$

The kinetic energy (per wavelength and per unit width of wavefront) is identical to the potential energy. So the true power of this model shallow-water tidal wave is

$$\text{Power} = \frac{1}{2}(\rho g h^2 \lambda) \times w/T = \frac{1}{2}\rho g h^2 v \times w, \tag{5}$$

where w is the width of the wavefront. Substituting $v = \sqrt{gd}$,

$$\text{Power} = \rho g h^2 \sqrt{gd} \times w/2. = \rho g^{3/2} \sqrt{d} h^2 \times w/2. \tag{6}$$

Let's compare this power with the kinetic-energy flux K_{BV}. Strikingly, the two expressions scale differently with amplitude. Using the amplitude conversion relation (2), the crest velocity (1), and $A = wd$, we can re-express the kinetic-energy flux as

$$K_{\mathrm{BV}} = \frac{1}{2}\rho A U^3 = \frac{1}{2}\rho w d (vh/d)^3 = \frac{1}{2}\rho w \left(g^{3/2}/\sqrt{d}\right) h^3. \tag{7}$$

Thus the kinetic-energy-flux method suggests that the total power of a shallow-water wave scales as amplitude cubed; but the correct formula shows that the power goes as amplitude squared.

The ratio is

$$\frac{K_{\rm BV}}{\rm Power} = \frac{\rho w \left(g^{3/2}/\sqrt{d}\right) h^3}{\rho g^{3/2} h^2 \sqrt{d} w} = \frac{h}{d} \tag{8}$$

Thus estimates based on the kinetic-energy-flux method may be too small by a significant factor, at least in cases where this shallow-water cartoon of tidal waves is appropriate.

Moreover, estimates based on the kinetic-energy-flux method incorrectly assert that the total available power at springs is greater than at neaps by a factor of eight (assuming an amplitude ratio, springs to neaps, of two); but the correct answer is that the total available power of a travelling wave scales as its amplitude squared, so the springs-to-neaps total-incoming-power ratio is four.

Intuition

Why is the kinetic-energy-flux method wrong for tidal waves in open shallow water? One way to think about it is to make an analogy with other processes where a moving body delivers energy to another body. If I grab someone with one hand and shake them around, how much power am I delivering? Can we find the power by putting a section through my arm and working out the kinetic-energy-flux of my arm? No! My arm might be made of balsa wood – that would completely change the kinetic-energy flux, but would not change the effect on the receiving body.

One reason people get confused about the power in a wave is because they think that the power moves at the same speed as the water. There are a few ways to see that this is not generally true: note the speed at which high tides move up the English channel or down the North Sea – they move at hundreds of miles per hour, while the water itself moves only at one or two miles per hour. Another thought experiment is to imagine a travelling transverse wave, where there is no motion at all in the direction of travel; in this case it is particularly clear that the kinetic energy method gives an incorrect answer.

The tidal wave conveys energy not because a piece of water moves along, carrying that energy with it as kinetic energy, but because the weight of water in a tidal peak exerts a pressure on neighbouring water, and that pressure does work as the water moves.

We can compute the power using this idea of the body of water on the left doing work on the body of water on the right.

Power flux using forces

Let's repeat the power calculation using forces. Consider a piston pressing against a wall of water, behaving just as an adjacent body of water would. During the half period when the piston moves to the right, while a crest is present, the work done on the piston at depth z is of order $P^+(z)UT$ per unit area, where $P^+(z)$ is the pressure at depth z, U is the velocity and T is the period. As the crest passes, the peak pressure is:

$$P^+(z) = \rho g(z+h). \tag{9}$$

When the piston moves to the left, the pressure is lower:

$$P^-(z) = \rho g(z-h); \tag{10}$$

and the work done on the piston is $-P^-(z)UT$ per unit area (neglecting constants of order 1, as before). The net work done on the piston (per unit area) is

$$P^+(z)UT - P^-(z)UT = UT(\rho g(z+h) - \rho g(z-h)) = UT(\rho g(2h)), \tag{11}$$

independent of z. Integrating up over the area $A = wd$, the average power delivered to the piston is

$$\text{Power} = wdUT(\rho g(2h))/T = 2wvh\rho gh = 2w\rho g^{3/2}\sqrt{d}h^2. \tag{12}$$

In this expression, the factor of 2 is bogus: I should have done the integral. This answer agrees with the other outcome (6).

Reconciliation in a special case

The kinetic-energy-flux method is not always wrong. In the special case of tidal flow through a narrow cleft connecting two immense reservoirs, one of which is tidal and one of which is scarcely so, it gives the right answer. An example of such a cleft might by the Strait of Gibraltar, connecting the tidal Atlantic with the not-very-tidal Mediterranean.[1] Imagine that at high water there is a height drop of h between stationary waters on the two sides. Assuming vorticity-free flow from the high side up to the outlet, the velocity U of water at the outlet of the cleft (at any depth) can be estimated

[1] In fact, the Strait of Gibraltar is much more complicated, with density differences simultaneously driving a surface inflow and a deeper outflow Tejedor et al. [1999], Morozov et al. [2002].

by applying Bernoulli's formula along a streamline connecting that water to a virtually-stationary upstream origin:

$$\frac{1}{2}\rho U^2 = \rho g h \tag{13}$$

$$U = \sqrt{2gh} \tag{14}$$

The kinetic-energy flux is

$$\frac{1}{2}\rho A U^3 = \frac{1}{2}\rho A \sqrt{2gh}^3 = \rho A \sqrt{2} g^{3/2} h^{3/2} \tag{15}$$

The total power arriving can also be written down in terms of potential energy drop:

$$\rho g h U A = \rho g h \sqrt{2gh} A. \tag{16}$$

These two equations (15,16) agree.

The extractable power by stream-turbines in this situation has been shown by Garrett and Cummins [2005] to be roughly 0.22 times the total power (16).

Literature

The fact that the power in a tidal wave scales as amplitude-*squared* is present in the detailed model of Taylor [1920]. Taylor's motivation in writing his paper, coincidentally, was to correct an underestimation of tidal dissipation! He shows, assuming that the height and the current both vary sinusoidally, that the flux of power passing into the Irish Sea is

$$\frac{1}{2} g \rho U w d h \cos \frac{2\pi}{T}(T_1 - T_0), \tag{17}$$

where $U = 1.63\,\text{m/s}$ is the peak tidal flow, $d = 68\,\text{m}$ is the depth of the channel, $h = 1.45\,\text{m}$ is the average tidal height (the half-range) along the line of width $w = 80\,\text{km}$; T_1 is the time of high water, T is the period of lunar tides, and T_0 is the time of maximum current. U and h are both proportional to amplitude, hence amplitude-squared. Taylor concluded that a power of 64 GW flowed into the Irish Sea. His formula (17) agrees with equation (6). He also estimated that three-quarters of this power was dissipated in bottom-friction in the Irish Sea, and one quarter re-emerged in a wave travelling in the opposite direction. This counter-travelling wave causes the rate of progress of high tides along the coast to be different from the velocity of the tidal waves; the relationship between the two velocities depends on the phase difference between the two waves. Taylor also analysed the effect of the moon itself on the energy flow in the Irish Sea, finding its contribution to be small (less than 10% of the incoming flux above). He estimated that the average dissipation rate in the Irish Sea was 1.5 W/m^2 of sea floor. Taylor's analysis includes several independent tests and verifications of his model. Taylor suggests that a reasonable model of frictional dissipation in the Irish Sea (and for winds on Salisbury plain) is

$$\text{power} = k\rho v^3 \text{ (per unit area)},$$

with $k = 0.002$.

Flather [1976] built a detailed numerical model of the lunar tide, chopping the continental shelf around the British Isles into roughly a thousand square cells. Their friction model has mean dissipation

$$\text{power} = k\rho v^3 \text{ (per unit area)},$$

with $k = 0.0025 - -0.003$. Flather estimates that the total average power entering this region is 215 GW. According to his model, 180 GW enters the gap between France and Ireland. From Northern Ireland round to Shetland, the incoming power is 49 GW. Between Shetland and Norway there is a net loss of 5 GW.

Measurements made over ten years near the edge of the continental shelf by Cartwright et al. [1980] verified and improved on Flather's estimates. Their experiments indicate that the average M2 power transmission was 60 GW between Malin Head (Eire) and Floro (Norway) and 190 GW between Valentia (Eire) and the Brittany coast near Ouessant. The power entering the Irish Sea was found to be 45 GW, and entering the North Sea via the Dover Straits, 16.7 GW. Near the Orkneys the incoming powers are 14 GW and 12 GW. They try to estimate the loss through bottom friction too (using $k = 0.0025$) and they estimate that there is less dissipation in the North Sea and Scottish waters (40 GW) than the incoming power (77 GW). They say they are not sure exactly where the correction to the loss arises. On a later page they mention finding that $k = 0.005$ is sometimes a better model.

Back to the shallow-water tidal wave model

Shelving

If the depth d decreases gradually and the width remains constant, I'd guess there'll be minimal reflection and the power of the wave will remain constant. This means $\sqrt{d}h^2$ is a constant, so we deduce that the height of the tide scales with depth as $h \sim 1/d^{1/4}$.

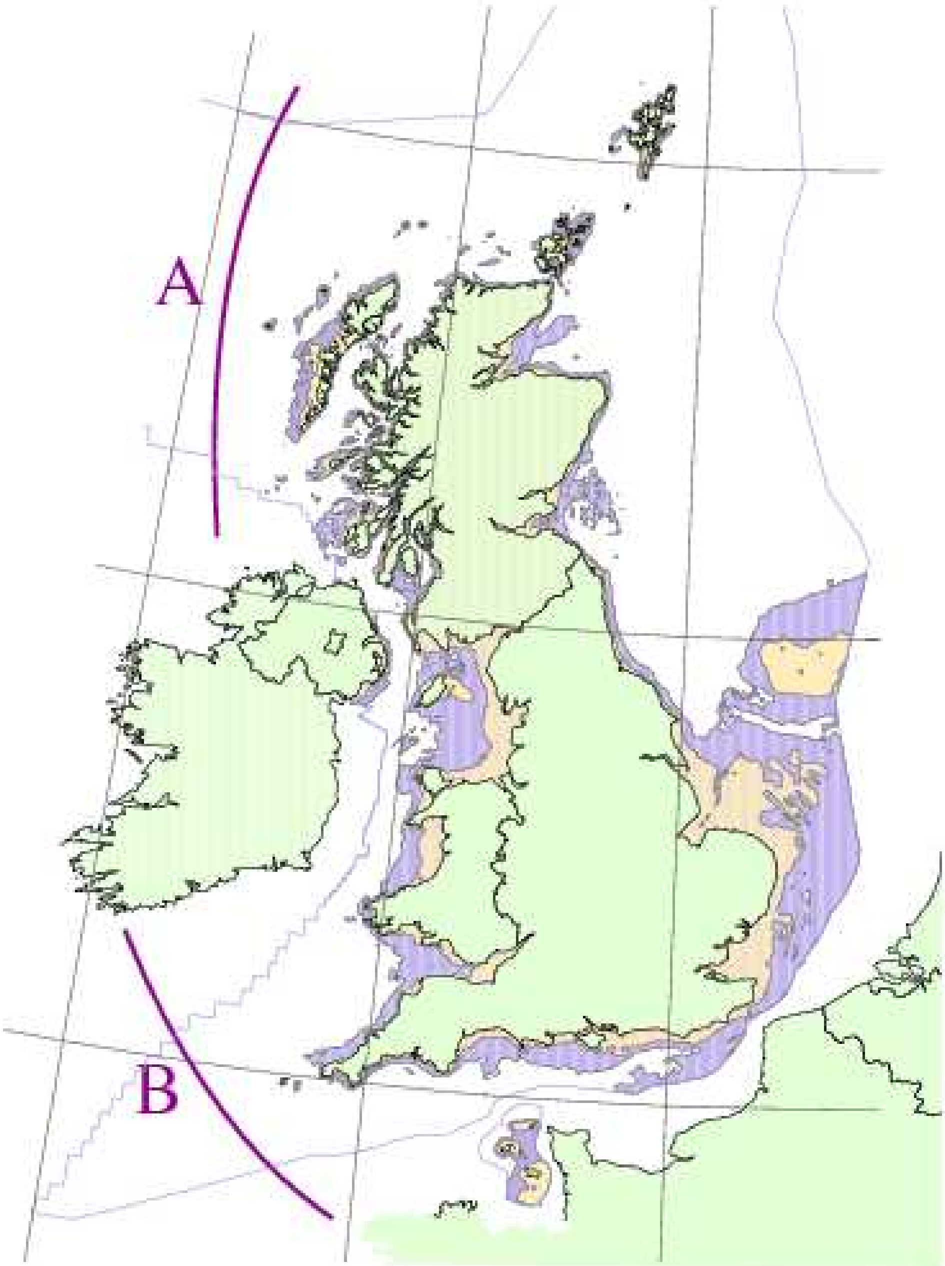

Figure 2. Two lines in the Atlantic. Bathymetry data from DTI Atlas of Renewable Marine Resources. © Crown copyright.

Application to the UK

Let's work out the power per unit length of wave crest for some plausible figures. If $d = 100\,m$, and $h = 1$ or $2\,m$, the power per unit length of wave crest is

$$\rho g^{3/2}\sqrt{d}h^2/2 = 10000\times 3\times 10\times(1 \text{ or } 4)/2 = \begin{cases} 150 \;\; \text{kW/m} & (h = 1\,\text{m}) \\ 600 \;\; \text{kW/m} & (h = 2\,\text{m}) \end{cases} \tag{18}$$

These figures are impressive compared with the raw power per unit length of Atlantic deep-water waves, 60–80 kW/m gross or 40–50 kW/m net [Mollison et al., 1976]. Clearly, the upper bound on tidal power is bigger by a factor of 10 or so than that for waves.

We can estimate the total incoming power from the Atlantic by multiplying appropriate lengths by powers-per-unit-length. My two lines A and B (figure 2) are both about 400 km long. The tidal range (at springs) on line A at depth $d = 100\,\text{m}$ is $2h = 3.5\,\text{m}$. For neaps, I'll assume a range of $2h = 1.8\,\text{m}$. The tidal range (at springs) on line B at depth $d = 100\,\text{m}$ is $2h = 4.5\,\text{m}$. At neaps on line B, $2h = 2.4\,\text{m}$ seems a reasonable estimate. Averaging the powers for springs and neaps, the incoming tidal resource over line A is 120 GW, and over line B, 195 GW. A total of 315 GW or 125 kWh per person per day. (Compare this with Salter's 100 GW (peak) dissipation in Pentland Firth.) In addition to lines A and B, I should perhaps include a line joining Shetland to Norway: much of the North Sea's tidal energy arrives across this line.[2] Let's throw in an extra 135 GW for the North Sea (an overestimate as I learned later from Cartwright et al. [1980]), making a round total of

$$450\,\text{GW} \tag{19}$$

or 180 kWh per person per day.

How much of this might conceivably be extracted? If we say 5%, and assume the conversion and transmission steps are 50% efficient, we arrive at

$$11\,\text{GW}, \tag{20}$$

or 4.5 kWh per person per day.

h (m)	$\rho g^{3/2}\sqrt{d}h^2/2$ (kW/m)
0.9	125
1.0	155
1.2	220
1.5	345
1.75	470
2.0	600
2.25	780

Figure 3. Power fluxes for depth $d = 100\,m$.

Acknowledgements

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[2] See animation at `http://www.math.uio.no/~bjorng/tidevannsmodeller/tidemod.html`

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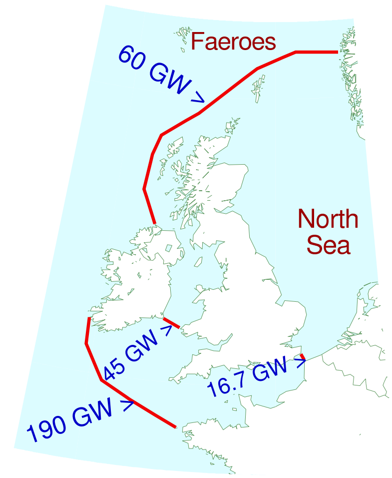
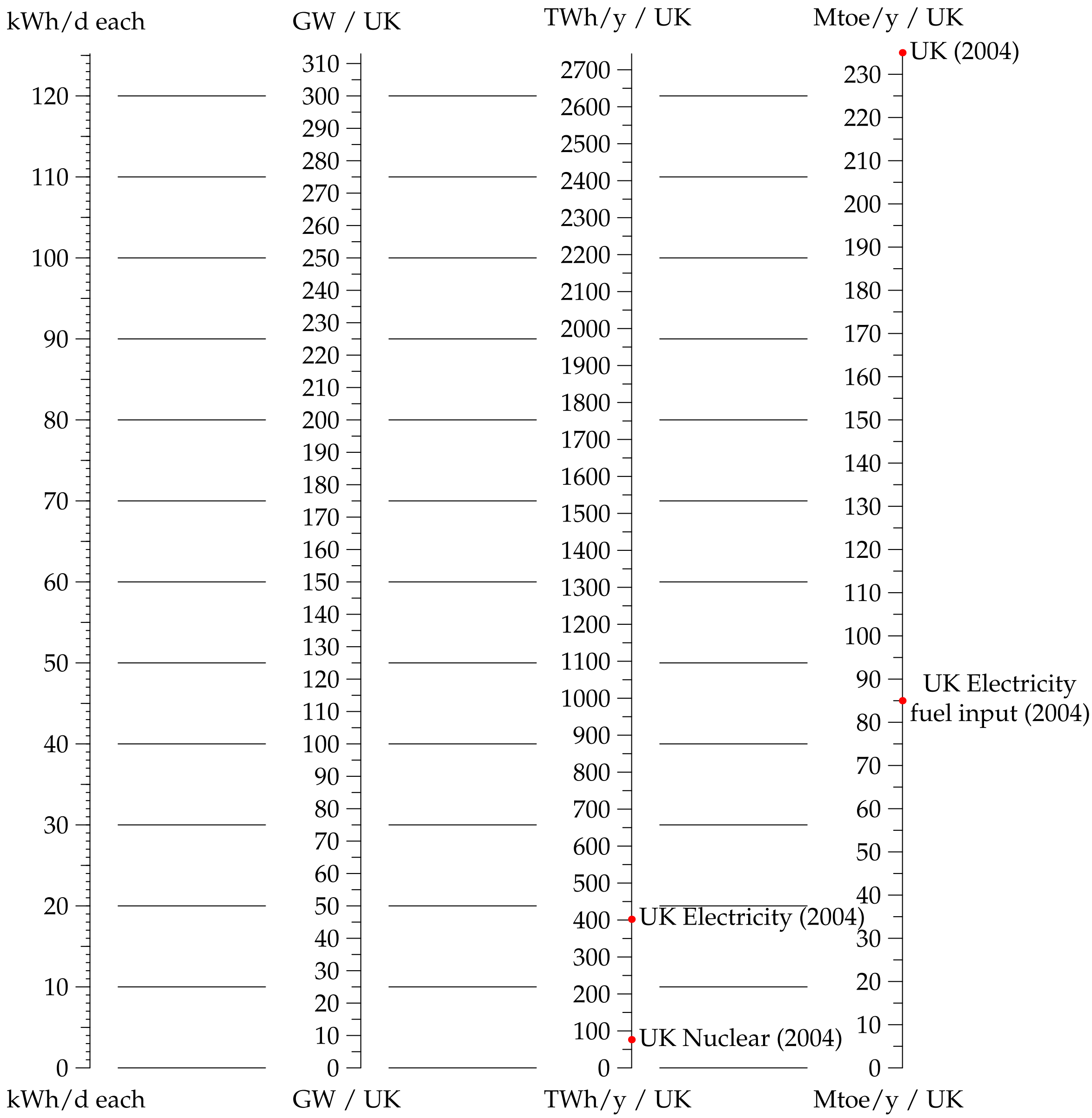


Figure 4. Average tidal powers measured by Cartwright et al. [1980].

Power translation chart



1 kWh/d	the same as 1/24 kW	'UK'	= 60 million people
GW	often used for '**capacity**' (peak output)		
TWh/y	often used for average output	USA:	300 kWh/d per person
1 Mtoe	'one million tonnes of oil equivalent'	Europe:	125 kWh/d per person