Since the length of the rod is \( l = R\theta \), the strain in the material at \( y \) is

\[
\epsilon = \frac{(R + y)\theta - R\theta}{R\theta} = \frac{y}{R} \tag{1}
\]

The energy density is

\[
dE = \frac{Yy^2}{2R^2}dV \tag{3}
\]

so the total stored energy in a beam is

\[
E = \frac{YIl}{2R^2} = \frac{YI\theta^2}{2l} \tag{4}
\]

where \( I \) is the moment of area

\[
I = \int y^2dA. \tag{6}
\]

Differentiating we can obtain the bending moment

\[
B = \frac{dE}{d\theta} = \frac{YI\theta}{l} = \frac{YI}{R}. \tag{9}
\]
The bending moment at $x$ due to a force at the end of the ruler is

$$B = F(l - x)$$

we also know that the beam obeys

$$B = \frac{YI}{R}$$

$$\approx YI \frac{d^2y}{dx^2}$$

where $I$ is the moment of area

$$I = \int y^2 dA$$

$$= \frac{ab^3}{12}.$$ 

Hence

$$YI \frac{d^2y}{dx^2} = F(l - x)$$

$$Y I_y = \frac{F}{6} (l - x)^3 + Ax + B$$

where $A$ and $B$ are integration constants which can be fixed using the boundary conditions that $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$ (since the rod is clamped vertically). Therefore

$$y = \frac{2F}{Ya} b^3 x^2 (3l - x).$$
4.3

The general form for the bending moment is

\[ B = \int_x^l (x' - x) f (x') \, dx' \]  

(18)

where \( f (x) \) is the force density. Therefore

\[ Y I \frac{d^2 y}{dx^2} = \int_x^l (x' - x) f (x') \, dx'. \]  

(19)

We can differentiate using the results

\[ \frac{dB}{dx} = -(x - x) f (x) - \int_x^l f (x') \, dx' \]  

(20)

\[ = - \int_x^l f (x') \, dx' \]  

(21)

\[ \frac{d^2 B}{dx^2} = f (x) \]  

(22)

to obtain

\[ Y I \frac{d^4 y}{dx^4} = f (x). \]  

(23)

We have the boundary conditions

• \( y = \frac{dy}{dx} = 0 \) at \( x = 0 \) since the beam is clamped.

• \( \frac{d^2 y}{dx^2} = \frac{d^2 y}{dx^3} = 0 \) from \( B = \frac{dB}{dx} = 0 \) (from equation 18)

\( f (x) \) is the force required to hold the beam static. When the beam is unsupported, \( f \) is the force that accelerates the beam

\[ \frac{d^2 y}{dt^2} dm = -f (x) \, dx \]  

(24)

\[ \frac{d^2 y}{dt^2} = -Y a b^3 12 \frac{d^4 y}{dx^4} \, dx \]  

(25)

\[ = \frac{Y b^2 d^4 y}{12 \rho \, dx^4} \]  

(26)

where the density is \( \rho \).
When solving, try a solution of the form
\[ y = y_0 \exp (i (kx - \omega t)) \]  
and therefore
\[ -\omega^2 + \frac{Y b^2}{12 \rho} k^4 = 0. \]  

Solutions are of the form \( k = \pm q \) and \( k = \pm iq \) where
\[ q = \sqrt{\omega \frac{12 \rho}{Y b^2}}. \]  

Hence a general form for the solution is
\[ y = A \cos qx + B \sin qx + C \cosh qx + D \sinh qx. \]

Applying the boundary conditions
\begin{itemize}
  \item \( y = 0 \) at \( x = 0 \) so \( A + C = 0 \)
  \item \( \frac{dy}{dx} = 0 \) at \( x = 0 \) so \( B + D = 0 \)
  \item \( \frac{d^2 y}{dx^2} = 0 \) at \( x = l \) so \(-A \cos ql - B \sin ql - A \cosh ql - B \sinh ql = 0 \)
  \item \( \frac{d^3 y}{dx^3} = 0 \) at \( x = l \) so \( A \sin ql - B \cos ql - A \sinh ql - B \cosh ql = 0 \)
\end{itemize}

Therefore
\[ A (\cos ql + \cosh ql) = -B (\sin ql + \sinh ql) \]  
\[ B (\cos ql + \cosh ql) = A (\sin ql - \sinh ql) \]  
and so
\[ (\cos ql + \cosh ql)^2 = - (\sin^2 ql - \sinh^2 ql) \]
\[ \cos^2 ql + \sin^2 ql + \cosh^2 ql - \sinh^2 ql + 2 \cos ql \cosh ql = 0 \]
\[ \cos ql \cosh ql = -1 \]

So if we define \( \alpha \) to be a solution of \( \cos \alpha \cosh \alpha = -1 \) then
\[ q = \frac{\alpha}{l} \]
\[ \omega = \frac{\alpha^2 b}{2l^2} \sqrt{\frac{Y}{3\rho}} \]