Contents:

Normal modes for the general equation $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$

Recap

In the first two lectures, we studied the special case where $\mathbf{M} = m\mathbf{1}$, so that the equation of motion is

$$\ddot{\mathbf{x}} = -\mathbf{A}\mathbf{x}, \tag{1}$$

where $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$ is a symmetric matrix. The eigenvectors of the symmetric matrix $\mathbf{A}$ are the solutions of

$$\mathbf{A}\mathbf{e}^{(a)} = \lambda^{(a)}\mathbf{e}^{(a)}, \tag{2}$$

which can also be written

$$\left[\mathbf{A} - \lambda^{(a)}\mathbf{1}\right]\mathbf{e}^{(a)} = 0. \tag{3}$$

The eigenvectors of the symmetric matrix $\mathbf{A}$ have the property that

> eigenvectors with different eigenvalue are orthogonal
> — *i.e.*, if $\lambda^{(a)} \neq \lambda^{(b)}$ then $\mathbf{e}^{(a)\mathsf{T}}\mathbf{e}^{(b)} = 0$.

We introduced a complete set of orthonormal eigenvectors of $\mathbf{A}$, $\{\mathbf{e}^{(a)}\}$, with eigenvalues $\{\lambda^{(a)}\}$, and we found that the general solution to the equation of motion (1) is a superposition of the normal modes:

$$\mathbf{x}(t) = \sum_a \mathbf{e}^{(a)} C_a \cos(\omega_a t + \phi_a), \tag{4}$$

where $\omega_a^2 = \lambda_a$, and $\{C_a\}$ and $\{\phi_a\}$ are the $2N$ arbitrary constants determined by the boundary conditions.

So, what changes when we upgrade to the more general equation of motion,

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x} \tag{5}$$

where $\mathbf{M}$ and $\mathbf{K}$ are symmetric matrices?

Summary

1. There are still N normal modes. Each is associated with a *generalized eigenvector*, which is a solution of

   $$\mathbf{K}\mathbf{e}^{(a)} = \lambda^{(a)}\mathbf{M}\mathbf{e}^{(a)}, \tag{6}$$

   which can also be written

   $$\left[\mathbf{K} - \lambda^{(a)}\mathbf{M}\right]\mathbf{e}^{(a)} = 0. \tag{7}$$

   The essential difference between this and the earlier eigenvector definition (3) is that the identity matrix $\mathbf{1}$ has been replaced by $\mathbf{M}$.

2. The general solution to the equation of motion (5) is, just as before, a superposition of the normal modes:

   $$\mathbf{x}(t) = \sum_a \mathbf{e}^{(a)} C_a \cos(\omega_a t + \phi_a), \tag{8}$$

   where $\omega_a^2 = \lambda_a$.

3. The only difference produced by the change of equation of motion from (1) to (5) is a change in the orthogonality properties of the normal modes. Whereas the orthogonality property for a symmetric matrix $\mathbf{A}$ is 'if $\lambda^{(a)} \neq \lambda^{(b)}$ then $\mathbf{e}^{(a)\mathsf{T}}\mathbf{e}^{(b)} = 0$', the orthogonality property for the generalized eigenvectors is:

   if $\lambda^{(a)} \neq \lambda^{(b)}$ then

$$\mathbf{e}^{(a)\mathsf{T}}\mathbf{M}\mathbf{e}^{(b)} = 0, \quad \text{or, equivalently} \quad \mathbf{e}^{(a)\mathsf{T}}\mathbf{K}\mathbf{e}^{(b)} = 0. \tag{9}$$

Examples

You can see two examples worked out numerically in RHB 233–238. The case of the double pendulum is worked out elsewhere in this handout.

Some details

1. We first show that the normal modes are given by the generalized eigenvector definition (7). Assume that the system performs a periodic motion

$$\mathbf{x} = \mathbf{x}_0 \cos(\omega t), \tag{10}$$

   so that the acceleration is

$$\ddot{\mathbf{x}} = -\omega^2 \mathbf{x}_0 \cos(\omega t). \tag{11}$$

   Substituting these two expressions into the equation of motion (5) and rearranging, we have

$$\mathbf{K}\mathbf{x}_0 = \omega^2 \mathbf{M}\mathbf{x}_0, \tag{12}$$

   so $\mathbf{x}_0$ is a solution of the generalized eigenvector problem with $\lambda = \omega^2$.

   The brute-force solution of this generalized eigenvector problem is first to find the eigenvalues by solving the polynomial equation

$$|\mathbf{K} - \lambda \mathbf{M}| = 0, \tag{13}$$

   then for each $\lambda^{(a)}$, solve (7) for $\mathbf{e}^{(a)}$. The brute force method should never be tackled by hand for $N > 2$ – there is always a better way!

2. The fact that the general solution to the equation of motion (5) is the superposition of the normal modes (8) is straightforward to prove by substitution of the claimed solution into the equation of motion.

3. The proof that the generalized eigenvectors satisfy the generalized orthogonality rules (9) is a simple modification of the proof for ordinary eigenvectors. We pick two eigenvectors $\mathbf{e}^{(a)}$ and $\mathbf{e}^{(b)}$ that satisfy

$$\mathbf{K}\mathbf{e}^{(a)} = \lambda^{(a)}\mathbf{M}\mathbf{e}^{(a)} \tag{14}$$

   and

$$\mathbf{K}\mathbf{e}^{(b)} = \lambda^{(b)}\mathbf{M}\mathbf{e}^{(b)}, \tag{15}$$

   and we *left-multiply* equation (14) by $\mathbf{e}^{(b)\mathsf{T}}$ and (15) by $\mathbf{e}^{(a)\mathsf{T}}$:

$$\mathbf{e}^{(b)\mathsf{T}}\mathbf{K}\mathbf{e}^{(a)} = \lambda^{(a)}\mathbf{e}^{(b)\mathsf{T}}\mathbf{M}\mathbf{e}^{(a)} \tag{16}$$

$$\mathbf{e}^{(a)\mathsf{T}}\mathbf{K}\mathbf{e}^{(b)} = \lambda^{(b)}\mathbf{e}^{(a)\mathsf{T}}\mathbf{M}\mathbf{e}^{(b)}. \tag{17}$$

Because both matrices $\mathbf{K}$ and $\mathbf{M}$ are symmetric, the products $\mathbf{e}^{(b)\mathsf{T}}\mathbf{K}\mathbf{e}^{(a)}$ and $\mathbf{e}^{(a)\mathsf{T}}\mathbf{K}\mathbf{e}^{(b)}$ are equal, and similarly $\mathbf{e}^{(b)\mathsf{T}}\mathbf{M}\mathbf{e}^{(a)} = \mathbf{e}^{(a)\mathsf{T}}\mathbf{M}\mathbf{e}^{(b)}$. Thus, subtracting (17) from (16),

$$(\lambda^{(a)} - \lambda^{(b)})\mathbf{e}^{(a)\mathsf{T}}\mathbf{M}\mathbf{e}^{(b)} = 0, \tag{18}$$

which shows that

if $\lambda^{(a)} \neq \lambda^{(b)}$ then

$$\mathbf{e}^{(a)\mathsf{T}}\mathbf{M}\mathbf{e}^{(b)} = 0. \tag{19}$$

That the alternative orthogonality statement

$$\mathbf{e}^{(a)\mathsf{T}}\mathbf{K}\mathbf{e}^{(b)} = 0 \tag{20}$$

is also true is left as an exercise.

Some extra notes on orthogonality: One way of thinking about the new orthogonality rules (19,20) is this: we are using the eigenvectors as our basis vectors, but they are not orthogonal; when our basis vectors are not orthogonal, we need to distinguish between the original vector space, in which the displacement $\mathbf{x}$, the velocity $\dot{\mathbf{x}}$, the acceleration $\ddot{\mathbf{x}}$, and the eigenvectors are found, and the *dual space*, in which quantities like the conjugate momentum $\mathbf{p} = \mathbf{M}\dot{\mathbf{x}}$ and the generalized force $\mathbf{f} = \mathbf{K}\mathbf{x}$ live. The product $\mathbf{M}\mathbf{e}^{(b)}$ is the *dual* basis vector to the eigenvector $\mathbf{e}^{(b)}$. The rules of the game permit one to make inner products only between vectors that live in reciprocal spaces, for example, the product of $\mathbf{x}$ with $\mathbf{K}\mathbf{x}$. To form any other product – for example, of $\mathbf{x}$ with itself – is an error; an example that makes this clear is where the different components of $\mathbf{x}$ have different dimensions, for example $\mathbf{x} = (\delta r, \delta\theta)$, in which case the forbidden product $\mathbf{x}^{\mathsf{T}}\mathbf{x} = (\delta r)^2 + (\delta\theta)^2$ is the dimensionally illegal sum of a squared length and a squared angle. If you want to measure how big a displacement $\mathbf{x} = (\delta r, \delta\theta)$ is, you have to use an appropriate *metric*. In this example, the kinetic energy is $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 = \frac{1}{2}\dot{\mathbf{x}}\mathbf{M}\dot{\mathbf{x}}$, where the matrix $\mathbf{M}$ is

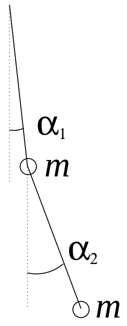
$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & mr^2 \end{bmatrix}, \tag{21}$$

so a natural way to measure the size of a displacement $\mathbf{x} = (\delta r, \delta\theta)$ is by the quadratic form $\mathbf{x}^{\mathsf{T}}\mathbf{M}\mathbf{x} = m(\delta r)^2 + mr^2(\delta\theta)^2$, which can be recognized as m times the squared Euclidean distance.

Displacement space
$\mathbf{x}$, $\dot{\mathbf{x}}$, $\ddot{\mathbf{x}}$ basis vectors: $\mathbf{e}^{(a)}$
Dual space
$\mathbf{p} = \mathbf{M}\dot{\mathbf{x}}$, $\mathbf{f} = \mathbf{K}\mathbf{x}$ dual basis vectors: $\mathbf{M}\mathbf{e}^{(a)}$, or $\mathbf{K}\mathbf{e}^{(a)}$

There is a general take-home message here: whenever someone measures how big a distance there is between two sets of numbers by summing the squares of the differences between the components, they are implicitly assuming that the appropriate metric is the identity matrix $\mathbf{1}$; it is often profitable to ask the question 'is this the right metric?'

Normal modes and the double pendulum



We find the equation of motion by Lagrangian methods. Because we are interested in the motion *near the fixed point* $(\alpha_1, \alpha_2) = (0, 0)$, we will *approximate* the Lagrangian, making an approximation accurate for small angles.

For small angles, the masses' kinetic energy is associated almost entirely with horizontal motion; the two horizontal speeds are approximately $l\dot{\alpha}_1$ and $l\dot{\alpha}_1 + l\dot{\alpha}_2 = l(\dot{\alpha}_1 + \dot{\alpha}_2)$. So the kinetic energy is

$$T \simeq \frac{1}{2}ml^2\dot{\alpha}_1^2 + \frac{1}{2}ml^2(\dot{\alpha}_1 + \dot{\alpha}_2)^2 = \frac{1}{2}ml^2\left[2\dot{\alpha}_1^2 + 2\dot{\alpha}_1\dot{\alpha}_2 + \dot{\alpha}_2^2\right]. \tag{22}$$

The potential energy is

$$\begin{aligned} V &= mgl(1-\cos\alpha_1) + mgl(1-\cos\alpha_1 + 1 - \cos\alpha_2) \\ &= 2mgl(1-\cos\alpha_1) + mgl(1-\cos\alpha_2). \end{aligned} \tag{23}$$

For small angles, we can use $\cos\alpha \simeq 1 - \frac{1}{2}\alpha^2 + \ldots$ to obtain

$$V \simeq 2mgl\frac{\alpha_1^2}{2} + mgl\frac{\alpha_2^2}{2}. \tag{24}$$

Notice that both these approximated energies can be written as quadratic forms:

$$T = \frac{1}{2}ml^2\begin{bmatrix}\dot{\alpha}_1 & \dot{\alpha}_2\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\dot{\alpha}_1\\ \dot{\alpha}_2\end{bmatrix}. \tag{25}$$

$$V = \frac{1}{2}mgl\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}. \tag{26}$$

Before going too much further, we should sanity-check these expressions. If $\alpha_1 = 0$ then $V = \frac{1}{2}mgl\alpha_2^2$ and $T = \frac{1}{2}ml^2\dot{\alpha}_2^2$, just like a simple pendulum. If $\alpha_2 = 0$, then the pendulum should have the same energies as a simple pendulum with mass $2m$; it does. And if $\alpha_1 = \alpha_2$ then the kinetic energy should be $\frac{1}{2}ml^2\dot{\alpha}_1^2 + \frac{1}{2}m(2l)^2\dot{\alpha}_1^2 = \frac{1}{2}ml^2 5\dot{\alpha}_1^2$. Good.

So now, what is the equation of motion?

$$L = T - V \simeq \frac{1}{2}ml^2\left[2\dot{\alpha}_1^2 + 2\dot{\alpha}_1\dot{\alpha}_2 + \dot{\alpha}_2^2\right] - \frac{1}{2}mgl\left[2\alpha_1^2 + \alpha_2^2\right]. \tag{27}$$

The conjugate momenta are

$$p_1 = \frac{\partial L}{\partial\dot{\alpha}_1} = 2ml^2\dot{\alpha}_1 + ml^2\dot{\alpha}_2 \tag{28}$$

$$p_2 = \frac{\partial L}{\partial\dot{\alpha}_2} = ml^2\dot{\alpha}_1 + ml^2\dot{\alpha}_2 \tag{29}$$

So the equations of motion are

$$\frac{dp_1}{dt} = \frac{\partial L}{\partial\alpha_1} = -2mgl\,\alpha_1 \tag{30}$$

$$\frac{dp_2}{dt} = \frac{\partial L}{\partial\alpha_2} = -mgl\,\alpha_2, \tag{31}$$

that is,

$$\begin{aligned} 2ml^2\ddot{\alpha}_1 + ml^2\ddot{\alpha}_2 &= -2mgl\,\alpha_1 && (32)\\ ml^2\ddot{\alpha}_1 + ml^2\ddot{\alpha}_2 &= -mgl\,\alpha_2. && (33) \end{aligned}$$

This equation of motion has the form $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$:

$$\mathbf{M}\begin{bmatrix} \ddot{\alpha}_1 \\ \ddot{\alpha}_2 \end{bmatrix} = -\mathbf{K}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \tag{34}$$

where

$$\mathbf{M} = ml^2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = mgl\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}. \tag{35}$$

How general is $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$?

Why is it claimed that virtually all dynamical systems have $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$ as their equation of motion near a fixed point? Let's take a general view of what we just did with the double pendulum, but consider a general system with coordinates $\mathbf{q}$, having a fixed point at $\mathbf{q} = \mathbf{q}^0$, $\dot{\mathbf{q}} = 0$.

What we did was, we made a Taylor series expansion of the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ about the fixed point $(\mathbf{q}, \dot{\mathbf{q}}) = (\mathbf{q}_0, 0)$.

$$L(\mathbf{q}, \dot{\mathbf{q}}) = L^{(0)} + \sum_i a_i \times (q_i - q_i^0) + \sum_i b_i \times \dot{q}_i + \ldots \tag{36}$$

Recall that for scalars, the Taylor expansion of $f(x)$ about x_0 is

$$f(x) = f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0}(x - x_0) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0}(x - x_0)^2 + \ldots;$$

For a function with two arguments $f(x, y)$,

$$\begin{aligned} f(x,y) &= f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{x_0,y_0}(y - y_0) + \\ &\quad \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0,y_0}(x - x_0)^2 + \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x_0,y_0}(y - y_0)(x - x_0) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial y^2}\right|_{x_0,y_0}(y - y_0)^2 + \ldots \end{aligned}$$

The first term in the expansion is a constant $L(\mathbf{q}^0, 0)$. The next terms are terms linear in the displacements $(q_i - q_i^0)$ and the velocities $\dot{q}_i$, with coefficients

$$a_i = \left.\frac{\partial L}{\partial q_i}\right|_{(\mathbf{q},\dot{\mathbf{q}})=(\mathbf{q}_0,0)} \tag{37}$$

and

$$b_i = \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{(\mathbf{q},\dot{\mathbf{q}})=(\mathbf{q}_0,0)}, \tag{38}$$

respectively. Since the state $(\mathbf{q}, \dot{\mathbf{q}}) = (\mathbf{q}_0, 0)$ is a fixed point, we expect the generalized forces $\frac{\partial L}{\partial q_i}$ all to be zero there. And typically, the conjugate momenta $\frac{\partial L}{\partial \dot{q}_i}$ are also zero at fixed points. So the linear terms are all zero.

This brings us to quadratic terms. It is very often the case that the quadratic terms can be written

$$\frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\mathbf{M}\dot{\mathbf{q}} - \frac{1}{2}(\mathbf{q} - \mathbf{q}^0)^{\mathsf{T}}\mathbf{K}(\mathbf{q} - \mathbf{q}^0). \tag{39}$$

This is not quite the most general possible form for the quadratic term in the Taylor expansion: there could be cross-terms coupling $\dot{\mathbf{q}}$ to $(\mathbf{q}-\mathbf{q}^0)$; but typically there are not. As we mentioned in lectures, matrices in quadratic forms such as $\mathbf{M}$ and $\mathbf{K}$ can always be chosen to be symmetric.

So, what have we found? Let's define the displacement from the fixed point $\mathbf{x} = \mathbf{q} - \mathbf{q}^0$. Then we have found that the Lagrangian is, to leading order,

$$L = L^{(0)} + \frac{1}{2}\dot{\mathbf{x}}^{\mathsf{T}}\mathbf{M}\dot{\mathbf{x}} - \frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{K}\mathbf{x}\ldots. \tag{40}$$

We differentiate to find the conjugate momenta, which we can write as a vector $\mathbf{p}$:

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \mathbf{M}\dot{\mathbf{x}}, \tag{41}$$

and the generalized forces, which make up a vector $\mathbf{f}$:

$$\mathbf{f} = \frac{\partial L}{\partial \mathbf{x}} = -\mathbf{K}\mathbf{x}. \tag{42}$$

Thus the equation of motion is

$$\frac{d}{dt}\mathbf{M}\dot{\mathbf{x}} = -\mathbf{K}\mathbf{x}, \tag{43}$$

that is,

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}. \tag{44}$$

Normal modes of the double pendulum

We can find the normal modes for

$$\mathbf{M} = \frac{1}{2}ml^2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = mgl\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{45}$$

by brute force. We first find the eigenvalues by solving

$$|\mathbf{K} - \lambda\mathbf{M}| = 0, \quad \text{or} \quad \begin{vmatrix} 2-2\lambda & -\lambda \\ -\lambda & 1-\lambda \end{vmatrix} = 0, \tag{46}$$

where, to save ink, we've omitted the factors of ml^2 and mgl.

$$2(1-\lambda)^2 - \lambda^2 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 2 = 0 \quad \Rightarrow \quad \lambda = 2 \pm \sqrt{2} \tag{47}$$

Thus the frequencies of the two normal modes are given by $\omega = \lambda^{1/2} = \sqrt{\left(2 \pm \sqrt{2}\right) g/l}$, that is, $1.8\sqrt{g/l}$ and $0.8\sqrt{g/l}$. We now find the corresponding displacements by solving

$$\begin{bmatrix} 2-2\lambda & -\lambda \\ -\lambda & 1-\lambda \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0. \tag{48}$$

All we are after is the ratio of e_1 to e_2, so we need only one of these two equations. We pick the bottom one.

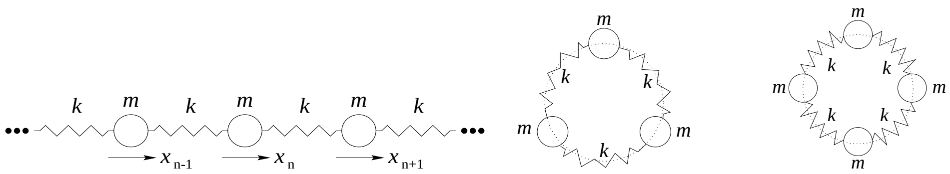
$$-(2 \pm \sqrt{2})e_1 + (1 - (2 \pm \sqrt{2}))e_2 = 0 \tag{49}$$

$$\Rightarrow \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \propto \begin{bmatrix} (-1 \mp \sqrt{2}) \\ (2 \pm \sqrt{2}) \end{bmatrix} = \begin{bmatrix} -2.4 \\ 3.4 \end{bmatrix} \text{ and } \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \tag{50}$$

Notice that these two eigenvectors are not orthogonal. You might check that they do satisfy the generalized orthogonality rule.

Eigenvectors of translation-invariant systems

A chain of identical masses connected by identical springs is a translation-invariant system if (a) the chain is infinitely long, or (b) the boundary conditions are periodic.



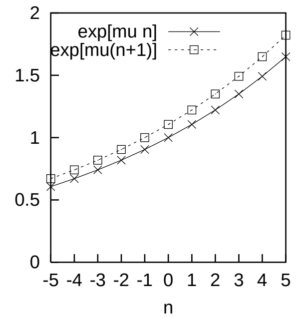
A symmetry operator $\mathbf{S}$ describing this symmetry under translation through one unit of distance is:

$$\mathbf{y} = \mathbf{S}\mathbf{x} \quad \text{where} \quad y_n = x_{n+1} \tag{51}$$

In the case of a 4×4 periodic system, the matrices $\mathbf{S}$ and $\mathbf{M}^{-1}\mathbf{K}$ are

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}^{-1}\mathbf{K} = \frac{k}{m} \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}. \tag{52}$$

If we can find the eigenvectors of $\mathbf{S}$, we will have found the eigenvectors of any $\mathbf{M}^{-1}\mathbf{K}$ that commutes with $\mathbf{S}$. So, what are these functions of n that look the same (except for a change of scale) when shifted to the left by one unit? Let's discuss the infinite system first.



The functions $e^{\mu n}$ and $e^{\mu(n+1)}$, for $\mu = 0.1$.

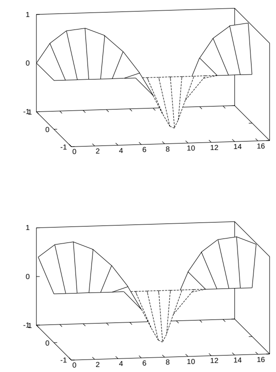
Infinite translation-invariant system

If $\mathbf{x}$ is an eigenvector with eigenvalue λ, *i.e.*,$\mathbf{S}\mathbf{x} = \lambda\mathbf{x}$, then (from the definition of $\mathbf{S}$, (51)) $x(n+1) = \lambda x(n)$. We can solve this relationship explicitly: $x(n) = \lambda^n x_0$, where x_0 is an arbitrary constant, which we can set to 1. Let's rewrite λ as $\lambda = e^{\mu}$. Then what we have found is this: the function $x(n) = e^{\mu n}$ is a function of n that rescales by a factor e^{μ} if it is shifted to the left:

$$y(n) = x(n+1) = e^{\mu(n+1)} = e^{\mu}e^{\mu n} = e^{\mu}x(n). \tag{53}$$

So $e^{\mu n}$ is an eigenvector of S with eigenvalue $\lambda = e^{\mu}$.

We are free to choose μ to be any real or complex value. If we define $\mu = i\kappa$ then $x(n) = e^{i\kappa n}$ is an eigenvector of S with eigenvalue $\lambda = e^{i\kappa}$.



(a) The function $e^{i\kappa n}$ as a function of n, for $\kappa = 2\pi/14$; (b) the same function, multiplied by $e^{i\kappa}$.

To visualize this function, think of a long spiral corkscrew. The long axis is the n-axis; the other two dimensions are the real and imaginary part of $x(n)$. If you rotate the corkscrew through an angle κ, the helix of the corkscrew appears to move along the long axis. So translation through some distance is equivalent to rotation through some angle. And rotation in the complex plane corresponds to multiplication by a complex number with unit magnitude, $e^{i\kappa}$.

Periodic translation-invariant system

There are a couple of ways of tackling the periodic system of length N. One is to think of it as an infinite system with the additional constraint that all functions must be periodic with period N.

This constraint restricts the allowed values of $\mu = i\kappa$. The eigenvectors of $\mathbf{S}$ have the form

$$x(n) = e^{i\kappa n}, \tag{54}$$

with eigenvalue $\lambda = e^{i\kappa}$, where the periodicity constraint $x(n) = x(n+N)$ implies

$$\kappa N = 2a\pi, \tag{55}$$

where a is an integer. The complete set of N eigenvectors is given by $a \in \{0, 1, 2, \ldots N-1\}$.

A second way to find the eigenvalues is to solve the equation

$$\det(\mathbf{S} - \lambda\mathbf{1}) = 0. \tag{56}$$

The determinant in the case $N = 4$ is

$$|\mathbf{S} - \lambda\mathbf{1}| = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = \lambda^4 - 1, \tag{57}$$

so the eigenvalues are the solutions of

$$\lambda^4 = 1, \tag{58}$$

which are the four fourth-roots of unity, $\lambda = \{1, e^{i\pi/2}, e^{2i\pi/2}, e^{3i\pi/2}\} = \{1, i, -1, -i\}$.

This result generalizes to the $N \times N$ case. The eigenvalues are always the solutions of

$$\lambda^N = 1, \tag{59}$$

and the eigenvectors $\mathbf{f}^{(a)}$ can be written $f_n^{(a)} = e^{i2\pi an/N}$. The transformations to and from the eigenvector basis are the discrete versions of the Fourier transform and the inverse Fourier transform.

The eigenvectors of the 4×4 matrix $\mathbf{S}$ are:

$$\begin{array}{c|cccc} \lambda & 1 & i & -1 & -i \\ \hline & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} & \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} \end{array} \tag{60}$$

And these must therefore be the normal modes of the circular four-mass system on the previous page. *Hang on,* you ask, *aren't the eigenvectors of a symmetric matrix real?* Yes, you can always find a complete set of real eigenvectors; but you don't have to! Here the eigenvectors that respect the rotation symmetry are not real; two of them are complex. You can check that they *are* eigenvectors of the matrix $\mathbf{M}^{-1}\mathbf{K}$ corresponding to the four-mass system. You will find that the two complex eigenvectors have the same eigenvalue. If you add and subtract these two eigenvectors, you can obtain *real* vectors that are eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$ with that same eigenvalue, but they will no longer be eigenvectors of $\mathbf{S}$. So you can stick to real eigenvectors if you want, but you will have to break the symmetry of the system to do so. The complex eigenvectors describe complex travelling waves; when you add them to make real eigenvectors, you can make either standing or travelling waves with the same frequency and wavelength.

DJCM. November 5, 2001