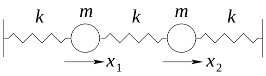
General motion of the symmetrical two-mass system



The two-mass system shown has equation of motion

$$\begin{aligned} m\ddot{x}_1 &= -(2kx_1 - kx_2) \\ m\ddot{x}_2 &= -(-kx_1 + 2kx_2), \end{aligned} \tag{1}$$

or

$$\ddot{\mathbf{x}} = -\mathbf{A}\mathbf{x}, \tag{2}$$

where

$$\mathbf{A} = \mathbf{M}^{-1}\mathbf{K} = \frac{k}{m}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \tag{3}$$

The eigenvectors and eigenvalues of this matrix are (c.f. exercise M.1)

$$\mathbf{e}^{(1)} = (1,1), \text{ with eigenvalue } \lambda^{(1)} = k/m,$$
$$\text{and } \mathbf{e}^{(2)} = (1,-1), \text{ with eigenvalue } \lambda^{(2)} = 3k/m.$$

General solution of the equation of motion (1)

We make a change of variables, introducing

$$u_1 \equiv x_1 + x_2 \tag{4}$$

$$\text{and } u_2 \equiv x_2 - x_1. \tag{5}$$

We now use the equation of motion (1) to find $\ddot{u}_1$ and $\ddot{u}_2$.

$$\begin{aligned} \ddot{u}_1 &= \ddot{x}_1 + \ddot{x}_2 = -\tfrac{k}{m}\left[(2x_1 - x_2) + (-x_1 + 2x_2)\right] = -\tfrac{k}{m}\left[x_1 + x_2\right] \\ \ddot{u}_2 &= \ddot{x}_2 - \ddot{x}_1 = -\tfrac{k}{m}\left[(-x_1 + 2x_2) - (2x_1 - x_2)\right] = -\tfrac{k}{m}\left[3x_2 - 3x_1\right] \end{aligned}$$

$$\Rightarrow \begin{aligned} \ddot{u}_1 &= -\tfrac{k}{m}u_1 \\ \ddot{u}_2 &= -3\tfrac{k}{m}u_2, \end{aligned} \quad \text{or} \quad \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = -\begin{bmatrix} \frac{k}{m} & 0 \\ 0 & 3\frac{k}{m} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{6}$$

The solutions to these uncoupled equations of motion are

$$\begin{aligned} u_1(t) &= C_1\cos(\omega_1 t + \phi_1) \\ u_2(t) &= C_2\cos(\omega_2 t + \phi_2) \end{aligned}, \tag{7}$$

where the angular frequencies are $\omega_1^2 = \frac{k}{m}$ and $\omega_2^2 = 3\frac{k}{m}$, and C_1, C_2, ϕ_1, and ϕ_2 are the free parameters of this general solution, which are determined by the initial conditions x_1, x_2, $\dot{x}_1$, and $\dot{x}_2$.

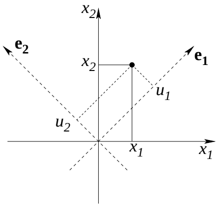
We now recover the original variables x_1 and x_2. Using

$$u_1 - u_2 = 2x_1 \quad \text{and} \quad u_1 + u_2 = 2x_2, \tag{8}$$

$$\begin{aligned} x_1(t) &= \tfrac{C_1}{2}\cos(\omega_1 t + \phi_1) - \tfrac{C_2}{2}\cos(\omega_2 t + \phi_2) \\ x_2(t) &= \tfrac{C_1}{2}\cos(\omega_1 t + \phi_1) + \tfrac{C_2}{2}\cos(\omega_2 t + \phi_2). \end{aligned} \tag{9}$$

Thus the general solution of the equation of motion is a superposition of the two normal modes

$$\begin{bmatrix} \cos(\omega_1 t + \phi_1) \\ \cos(\omega_1 t + \phi_1) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\cos(\omega_2 t + \phi_2) \\ +\cos(\omega_2 t + \phi_2) \end{bmatrix}. \tag{10}$$



Let us review the key steps made in finding the general solution. First, we introduced the new variables $u_1 = x_1 + x_2$ and $u_2 = x_2 - x_1$. These variables are the *projections* of $\mathbf{x} = (x_1, x_2)$ onto the two eigenvectors $\mathbf{e}^{(1)} = (1, 1)$ and $\mathbf{e}^{(2)} = (1, -1)$.

$$u_1 = \mathbf{e}^{(1)} \cdot \mathbf{x} \qquad u_2 = \mathbf{e}^{(2)} \cdot \mathbf{x} \tag{11}$$

This operation may be described as a change of basis. In the eigenvector basis, the variables are uncoupled, and the matrix $\mathbf{A}$ is transformed to a diagonal matrix (6).

To change back from the eigenvector basis (equation 8), we added up appropriately scaled basis vectors:

$$\mathbf{x} = u_1 \mathbf{e}_R^{(1)} + u_2 \mathbf{e}_R^{(2)}, \tag{12}$$

where

$$\mathbf{e}_R^{(1)} = \frac{1}{2}\mathbf{e}^{(1)} \;\text{ and }\; \mathbf{e}_R^{(2)} = \frac{1}{2}\mathbf{e}^{(2)} \tag{13}$$

are the *dual* vectors to $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ (also known as the *reciprocal basis*).

[If we'd used $\mathbf{e}^{(1)} = (1/\sqrt{2}, 1/\sqrt{2})$ and $\mathbf{e}^{(2)} = (1/\sqrt{2}, -1/\sqrt{2})$ as our eigenvectors, then the duals would equal the eigenvectors.]

We can use this viewpoint to describe the solution to other equations of the same form.

General solution of $\ddot{\mathbf{x}} = -\mathbf{A}\mathbf{x}$ for symmetric $\mathbf{A}$

Let the eigenvectors of $\mathbf{A}$, which satisfy

$$\mathbf{A}\mathbf{e}^{(a)} = \lambda^{(a)} \mathbf{e}^{(a)}, \quad \text{for } a = 1 \ldots N, \tag{14}$$

be normalized such that

$$\mathbf{e}^{(a)\mathsf{T}} \mathbf{e}^{(b)} = \delta_{ab}. \tag{15}$$

We now project $\mathbf{x}(t)$ onto the eigenvectors. $u_a(t)$ is the component of $\mathbf{x}(t)$ in direction $\mathbf{e}^{(a)}$:

$$u_a(t) = \mathbf{e}^{(a)\mathsf{T}} \mathbf{x}(t). \tag{16}$$

We *left-multiply* the equation of motion (2) by $\mathbf{e}^{(a)\mathsf{T}}$:

$$\mathbf{e}^{(a)\mathsf{T}} \ddot{\mathbf{x}} = -\mathbf{e}^{(a)\mathsf{T}} \mathbf{A}\mathbf{x}. \tag{17}$$

Now, $\mathbf{e}^{(a)\mathsf{T}} \mathbf{A} = \lambda^{(a)} \mathbf{e}^{(a)\mathsf{T}}$, so

$$\ddot{u}_a(t) = -\lambda^{(a)} \mathbf{e}^{(a)\mathsf{T}} \mathbf{x} \tag{18}$$

$$= -\lambda^{(a)} u_a(t). \tag{19}$$

So each of the projections $u_a(t)$ performs independent simple harmonic motion at frequency $\omega_a = \sqrt{\lambda^{(a)}}$.

We can reconstruct $\mathbf{x}(t)$ from its projections:

$$\mathbf{x}(t) = \sum_a \mathbf{e}^{(a)} u_a(t) = \sum_a \mathbf{e}^{(a)} C_a \cos(\omega_a t + \phi_a). \tag{20}$$

$\{C_a\}$ and $\{\phi_a\}$ are the $2N$ arbitrary constants determined by the boundary conditions – the initial positions and velocities.

So *the general solution to the equation of motion (2) is a superposition of the normal modes.*

DJCM. November 7, 2001