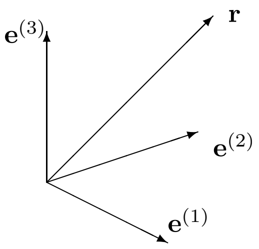


Rotating frames

Life in an inertial frame

Consider a point mass m whose location as a function of time is $\mathbf{r}(t)$. Let's express the location $\mathbf{r}$ in terms of three fixed basis vectors, $\mathbf{e}^{(i)}$.

$$\mathbf{r} = \sum_i r_i \mathbf{e}^{(i)}. \tag{1}$$

If we are in an inertial frame with fixed basis vectors, the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are defined by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \sum_i \dot{r}_i \mathbf{e}^{(i)} \tag{2}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \sum_i \ddot{r}_i \mathbf{e}^{(i)}. \tag{3}$$

The equation of motion of the mass, if subjected to a force $\mathbf{f}$, is

$$m\mathbf{a} = \mathbf{f}. \tag{4}$$

Life in a rotating frame

Fred chooses to represent $\mathbf{r}$, and all other vectors, using a set of basis vectors $\mathbf{e}^{(i)}$ that vary with time. For example, they might rotate at angular velocity $\underline{\omega}$. Fred can still define the components r_i of $\mathbf{r}$ as in (1), but to emphasize what is going on, let's show the time-dependence:

$$\mathbf{r}(t) = \sum_i r_i(t)\, \mathbf{e}^{(i)}(t). \tag{5}$$

As an example, imagine that the mass is actually stationary. In an inertial basis, all the components r_i would be constant; but if the basis vectors are time-varying, then the components $r_i(t)$ have to vary too, just to keep $\mathbf{r}$ in the same place.

Fred likes to pretend he is in an inertial frame, so he calls the quantity

$$\mathbf{v}_{\mathrm{R}} \equiv \sum_i \dot{r}_i(t)\, \mathbf{e}^{(i)}(t) \tag{6}$$

the 'velocity' of the particle and he calls

$$\mathbf{a}_{\mathrm{R}} \equiv \sum_i \ddot{r}_i(t)\, \mathbf{e}^{(i)}(t) \tag{7}$$

the 'acceleration'. In fact, these two vectors are not the true velocity and acceleration. That's why I've appended the subscript 'R' for 'rotating', to indicate the frame they are measured relative to.

Let's work out what the true velocity and acceleration are, starting from the expression for $\mathbf{r}$ (5). Differentiating with the help of the product rule,

$$\frac{d\mathbf{r}}{dt} = \sum_i \frac{dr_i}{dt}\mathbf{e}^{(i)} + \sum_i r_i \frac{d}{dt}\mathbf{e}^{(i)} \tag{8}$$

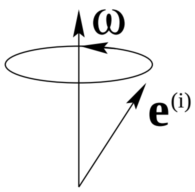
$$= \mathbf{v}_{\mathrm{R}} + \sum_i r_i \frac{d}{dt}\mathbf{e}^{(i)}. \tag{9}$$

$$\frac{d^2\mathbf{r}}{dt^2} = \sum_i \frac{d^2 r_i}{dt^2}\mathbf{e}^{(i)} + 2\sum_i \frac{dr_i}{dt}\frac{d}{dt}\mathbf{e}^{(i)} + \sum_i r_i \frac{d^2}{dt^2}\mathbf{e}^{(i)} \tag{10}$$

$$= \mathbf{a}_\mathrm{R} + 2\sum_i \frac{dr_i}{dt}\frac{d}{dt}\mathbf{e}^{(i)} + \sum_i r_i \frac{d^2}{dt^2}\mathbf{e}^{(i)}. \tag{11}$$

Now, if Fred insists on describing his world using the Newtonian laws that would apply in an inertial frame, what forces does he conclude are acting? Equation (11) gives the true acceleration, which must be equal to $\mathbf{f}/m$, where $\mathbf{f}$ is the sum of the true physical forces acting. But in his opinion, the 'acceleration' is $\mathbf{a}_\mathrm{R}$, where

$$\mathbf{a}_\mathrm{R} = \mathbf{f}/m - 2\sum_i \frac{dr_i}{dt}\frac{d}{dt}\mathbf{e}^{(i)} - \sum_i r_i \frac{d^2}{dt^2}\mathbf{e}^{(i)}. \tag{12}$$



So, the apparent 'acceleration' is the sum of $\mathbf{f}/m$ and two other terms.

Let us now evaluate these two terms for the important case of a steadily rotating basis.

Steady rotation

Consider one basis vector $\mathbf{e}^{(i)}$ rotating at angular velocity $\underline{\omega}$. What are $\frac{d}{dt}\mathbf{e}^{(i)}$ and $\frac{d^2}{dt^2}\mathbf{e}^{(i)}$?

$$\frac{d}{dt}\mathbf{e}^{(i)} = \underline{\omega} \times \mathbf{e}^{(i)} \tag{13}$$

Using the product rule, and the fact that the rotation rate is constant,

$$\frac{d}{dt}\left(\frac{d}{dt}\mathbf{e}^{(i)}\right) = \frac{d\underline{\omega}}{dt} \times \mathbf{e}^{(i)} + \underline{\omega} \times \frac{d}{dt}\mathbf{e}^{(i)} \tag{14}$$

$$= 0 + \underline{\omega} \times (\underline{\omega} \times \mathbf{e}^{(i)}). \tag{15}$$

We now substitute these identities into the 'acceleration' formula (12):

$$\mathbf{a}_\mathrm{R} = \mathbf{f}/m - 2\sum_i \frac{dr_i}{dt}\,\underline{\omega} \times \mathbf{e}^{(i)} - \sum_i r_i\,\underline{\omega} \times (\underline{\omega} \times \mathbf{e}^{(i)}) \tag{16}$$

$$= \mathbf{f}/m - 2\,\underline{\omega} \times \sum_i \frac{dr_i}{dt}\mathbf{e}^{(i)} - \underline{\omega} \times (\underline{\omega} \times \sum_i r_i \mathbf{e}^{(i)}) \tag{17}$$

$$= \mathbf{f}/m - 2\,\underline{\omega} \times \mathbf{v}_\mathrm{R} - \underline{\omega} \times (\underline{\omega} \times \mathbf{r}). \tag{18}$$

So, from Fred's point of view, the mass×'acceleration' is

$$m\mathbf{a}_\mathrm{R} = \mathbf{f} - 2m\,\underline{\omega} \times \mathbf{v}_\mathrm{R} - m\,\underline{\omega} \times (\underline{\omega} \times \mathbf{r}) \tag{19}$$

$$= \text{physical force} + \text{Coriolis force} + \text{centrifugal force}. \tag{20}$$

Conclusion

You can use Newtonian mechanics in rotating frames as long as you add the following forces to the physical forces acting:

$$\text{Coriolis force} = -2m\,\underline{\omega} \times \mathbf{v}_\mathrm{R} \tag{21}$$

$$\text{Centrifugal force} = -m\,\underline{\omega} \times (\underline{\omega} \times \mathbf{r}) \tag{22}$$

DJCM. November 15, 2001