Dimensional Analysis

The method:

1. List all variables and their dimensions.
2. Count the number of variables V and the number of independent dimensional constraints D.
3. Find $V - D$ independent dimensionless groups, $\theta_1, \theta_2, \theta_3, \ldots$, by multiplying variables.
4. The most general possible relationship among the variables is

$$\theta_1 = F(\theta_2, \theta_3, \ldots),$$

where F is a dimensionless function.

Queries:

Q1. What are 'independent dimensional constraints', and when is the number of them not equal to the number of dimensions appearing in the list of variables?

Q2. What are 'independent dimensionless groups'?

Q3. If there's more than one dimensionless group, does the answer given by dimensional analysis depend on guesswork?

Answers:

Two examples where the number of dimensional constraints is smaller than the apparent number of dimensions, 3 (that would be suggested by the usually-reliable rule of thumb):

(a) 'How does relativistic mass m depend on rest mass and speed?':

relativistic mass	m	M
rest mass	m_0	M
speed	v	LT^{-1}
speed of light	c	LT^{-1}
	4	only 2 constraints

Two dimensionless groups are m/m_0 and v/c, and the answer from dimensional analysis is $m = m_0 F(v/c)$.

(b) 'How does kinetic energy depend on mass and speed?':

mass	m	M
speed	v	LT^{-1}
kinetic energy	T	ML^2T^{-2}
	3	only 2 constraints

There is one dimensionless group, $T/(mv^2)$, and the answer from dimensional analysis is $T/(mv^2) = F()$; the function with no arguments is a constant, so $T = \kappa m v^2$, where κ is a dimensionless constant.

A dimensionless group θ formed from variables a, b, c, d is a product of the variables raised to powers $\alpha, \beta, \gamma, \delta$, such that:

- $\theta = a^\alpha b^\beta c^\gamma d^\delta$ is dimensionless.
- $(\alpha, \beta, \gamma, \delta) \neq (0, 0, 0, 0)$.

The constraint that θ be dimensionless implies **linear relationships** among $\alpha, \beta, \gamma, \delta$. For the relativistic mass example, the constraint that $\theta \equiv m^\alpha m_0^\beta v^\gamma c^\delta$ has no mass (M) dimension gives:

$$\alpha + \beta = 0; \tag{1}$$

the constraint that θ has no length (L) dimension gives:

$$\gamma + \delta = 0; \tag{2}$$

the third dimensional constraint on θ (no time T) reproduces constraint (2), so the number of independent constraints is just 2.

A set of dimensionless groups θ_1, θ_2, θ_3, are **independent** if there is no relationship of the form

$$\theta_l = \theta_m^\mu \theta_n^\nu$$

between them. Equivalently, if we think of the numbers $(\alpha, \beta, \gamma, \delta)$ associated with each group as defining vectors, the groups are independent if the vectors $(\alpha, \beta, \gamma, \delta)_1$, $(\alpha, \beta, \gamma, \delta)_2$, $(\alpha, \beta, \gamma, \delta)_3$ are linearly independent.

Freedom to guess dimensionless groups. If there's more than one dimensionless group, then there are indeed many choices for the set of independent groups. For example, the triangle problem

area	A	L^2
base	a	L
height	b	L
	3	1 constraint

can yield the dimensionless groups

$$\left(\frac{A}{a^2}\right) \quad \text{and} \quad \left(\frac{a}{b}\right);$$

or

$$\left(\frac{A}{ab}\right) \quad \text{and} \quad \left(\frac{b}{a}\right);$$

or

$$\left(\frac{Aa}{b^3}\right) \quad \text{and} \quad \left(\frac{Ab^2}{a^4}\right);$$

But the answer from dimensional analysis,

$$\text{(one group)} = F(\text{all the other groups}),$$

is always correct, and this answer is equivalent for any choice of the independent groups. All that changes is the meaning and form of the dimensionless function F. For example, for the first two choices of groups listed above, $F_1(x) = 1/(2x)$ and $F_2(x) = 1/2$ respectively. If we wish to express the answer from dimensional analysis in the form (one variable) $= f$(the others), for example $A = f(a, b)$ then we should ensure that the chosen variable appears in only one of the groups. So the choice of groups

$$\left(\frac{A}{ab}\right) \quad \text{and} \quad \left(\frac{A}{a^2}\right);$$

is valid, but not useful when we are trying to write $A = f(a, b)$.