# Handout 4

## Drag force, terminal velocity, and dimensional analysis

In lecture 7, we dropped a paper cone and asked 'how does terminal velocity $v_{\rm term}$ depend on weight, all other things being kept fixed?'

We recognised that the question could be rephrased as 'how does drag force $F$ depend on velocity $v$?' If $F \propto v$, then the terminal velocity would be proportional to the mass, $m$; if $F \propto v^2$, then the terminal velocity would go as $m^{1/2}$.

The demonstration appeared to show that the terminal velocity goes as $m^{1/2}$. What can we deduce from this result?

Let's use dimensional analysis to determine how the drag force ($F$) might depend on viscosity ($\eta$), density ($\rho$), speed ($v$), and size ($a$). [Note that we are not talking about *terminal* velocity any more; we're talking about the dependence of drag $F$ on a general velocity $v$.]

| variable | dimension |
|---|---|
| $F$ | $MLT^{-2}$ |
| $\eta$ | $ML^{-1}T^{-1}$ |
| $\rho$ | $ML^{-3}$ |
| $v$ | $LT^{-1}$ |
| $a$ | $L$ |
| 5 | 3 |

We need to find $5-3=2$ independent dimensionless groups.

We might choose the following two groups:

$$\Pi_1 \equiv \frac{F}{\rho v^2 a^2},$$

$$\Pi_2 \equiv \frac{av\rho}{\eta}.$$

I deliberately chose to make the second group have no $F$ in it, so that when we write down the general conclusion

$$\Pi_1 = G(\Pi_2),$$

where $G$ is a dimensionless function, we can easily rearrange this conclusion in the form with $F$ on the left hand side only:

$$F = \rho v^2 a^2 \, G\!\left(\frac{av\rho}{\eta}\right). \tag{1}$$

[As an alternative first group, I could have chosen the product of the groups $\Pi_1$ and $\Pi_2$:

$$\Pi_1' \equiv \Pi_1 \Pi_2 = \frac{F}{va\eta}.$$

This choice would lead to the equivalent conclusion

$$F = \eta v a \, G'\!\left(\frac{av\rho}{\eta}\right), \tag{2}$$

where $G'$ is a dimensionless function.]

Now, if we establish that $F \propto v^2$ over a wide range of velocities – our single experiment in the lecture doesn't take us quite that far, but let's make the assumption! – what can we deduce about the drag force? Well, if $F \propto v^2$, then the function $G$ in equation (1) must be a constant, so *the drag force must be independent of viscosity!*

This idea is quite startling when you first encounter it. The explanation is that the drag is caused not by viscous dissipation but by *turbulence*.

You'll learn more about drag and fluid flow in the Fluids course next year. The dimensionless group $\left(\frac{av\rho}{\eta}\right)$ is called the Reynolds' number of the flow. The function $G\!\left(\frac{av\rho}{\eta}\right)$ in equation (1) is not actually constant for *all* velocities: at sufficiently small velocities, when the Reynolds' number is less than about 1, the viscous forces (which were so tiny as to be negligible in our experiment) become significant, and a new dependence of $F$ on $v$ takes over [guess what it is!]. Viscous forces smooth out velocity gradients in a flow (think of honey) and thus prevent turbulence.

The Reynolds' number is a dimensionless measure of how significant viscous forces are in the situation. The Reynolds' number for the lecture demonstration was about 5000: huge Reynolds' numbers mean that viscosity is small.