Part IB Advanced Physics

# Dynamics

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The course webpage is
`http://wol.ra.phy.cam.ac.uk/teaching/dynamics/`.

Questions about the course are answered on the webpage. Also, I will hold a clinic after lectures in the Old Bursary, Darwin College.

## Handout 1

This handout contains 5 collections of examples:

1. **Maths recap.** (Page 3.) Exercises, with solutions, to help you check that you know the 1A maths material used in this course. This section also includes two questions (M.9,10) on new Physics that is based on this mathematical material.
2. **Traditional problems.** (Page 13.) Traditional problems of the sort found in examinations.
3. **Quickies.** (Page 20.) Some shorter problems involving order of magnitude Physics and dimensional analysis.
4. **Deep thought.** (Page 21.) A collection of puzzles and paradoxes to help you develop critical thinking skills, pull apart poor physical models, and replace them with good ones.
5. **Lecture problems.** Examples that will be worked through in lectures. (Page 24.)

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Worked solutions to most exercises will be put on the course webpage, but you are encouraged not to look at them before having a good go yourself.

I recommend two textbooks:

> Hand and Finch (1998). *Analytical Mechanics.* Cambridge University Press.
>
> Kibble and Berkshire (1985, 1996). *Classical Mechanics.* Addison Wesley Longman.

Hand and Finch is the most interesting book, and will take you up to the level of part II Theoretical Physics and beyond. Kibble and Berkshire is at the same level as the course.

# Advice

In all problems, you should apply the following skills:

**Estimate the answer before starting.** Guess. Don't be afraid to be wrong. Sketch a graph of the answer you are expecting. Why?

1. If you don't guess, and you later make a mistake in your answer, then you may fail to detect that your answer is silly.
2. If you get the answer right and your initial guess turns out to be wrong, your guessing machinery gets an opportunity to learn. You'll guess better next time.

**Draw a picture.** Just do it!

**Use dimensional analysis.** Identify the inputs and outputs in the situation, and see what inputs each output could depend on. Anticipate the relationships that could emerge. (Often dimensions give the complete answer, except for a dimensionless factor).

**Plan your campaign.** Count how many unknowns there are, and identify where the necessary constraints are going to come from.

**Use calculus with care.** Note all assumptions and approximations as you make them.

- When cancelling factors, could you be dividing by zero?
- Include constants of integration, and think what they mean.

**Sketch graphs.** Sketch a graph of every mathematical function you derive. Understand the meanings of the various terms.

**Write informative solutions.** Write as if writing an explanation to a colleague. Indicate your reasoning at every step. If you are not sure of a particular step, make a note mentioning your uncertainty.

**Sanity-check your answers.**

1. Is your answer dimensionally valid? [Substitute in numbers only at the last possible moment!]
2. Do your numerical answers fit with your guess?
3. Think about special cases. e.g., Does your solution have the right limiting behaviour as $m \to 0$, or $m \to \infty$, or $m/M = 1$?
4. Approximate your answer: under what conditions can some terms be negelected, and what happens to the answer under those conditions?

# 1 Maths recap

These exercises are provided, with worked solutions, to help you check that you are familiar with the 1A Maths material used by this course. This section also includes two questions (M.9,10) on **new Physics** that is based on this mathematical material.

If you need to read up on this material, you can find it in the following books:

**Revision of vector calculus.** Chapter 8 RHB (RHB 248–270).

Vector definition of torque and of angular momentum. Gradient of scalar field (RHB 263). Divergence and curl of a vector field.

**Tensors and Summation convention.** RHB 674–708 (Especially RHB 675&690) (Or HF 333–339 and HF 326–328 for tensors). Outer products RHB 683. The delta and epsilon tensors RHB 688–691. The vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Dual tensors RHB 696. Reciprocal bases RHB 704-705.

**Angular momentum and Kinetic Energy.** HF sec 8.1–8.3; HF 284–292. (Or RHB 697–699).
Definition of Moment of Inertia Tensor and Angular Momentum. Decomposition of kinetic energy of rigid body into a translation term and a rotation term.

**Matrices.** RHB Chapter 7 (RHB 184)

Eigenvectors and eigenvalues. (RHB 214)

**RHB** = Riley, Hobson and Bence, Mathematical Methods, CUP.

**HF** = Louis Hand and Janet Finch, Analaytical Mechanics, CUP.

## Eigenvectors and commuting matrices

## M.1

(a) Find the eigenvectors and eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

(b) Write $\mathbf{A}$ explicitly as the sum of two outer products, using the relationship

$$\mathbf{A} = \sum_a \lambda^{(a)} \mathbf{e}^{(a)} \mathbf{e}^{(a)^\mathsf{T}}, \tag{1}$$

where $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are eigenvectors with unit length. Check that your representation of $\mathbf{A}$ as this sum of two matrices works by computing the product $\mathbf{Ax}$ for $\mathbf{x} = (1, 0)$.

(c) Express $\mathbf{x}$ as a sum of the eigenvectors,

$$\mathbf{x} = \sum_a x_a \mathbf{e}^{(a)}. \tag{2}$$

(d) Find the eigenvectors and eigenvalues of the matrix $\mathbf{S} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Show that $\mathbf{A}$ and $\mathbf{S}$ commute (*i.e.*, that $\mathbf{AS} = \mathbf{SA}$).

(e) Find a complete set of eigenvectors and eigenvalues of the matrix $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Show that $\mathbf{A}$ and $\mathbf{1}$ commute. Are the eigenvectors of $\mathbf{A}$ eigenvectors of $\mathbf{1}$?

**M.2** Find three orthogonal eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

**M.3** Find the eigenvectors and eigenvalues of the matrix $\mathbf{S} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

# Summation convention

[See the solutions section (p. 6) for a summary of summation convention.]

## M.4

(a) Using the Einstein summation convention, show that the gradient of $\mathbf{x}^{\mathsf{T}}\mathbf{x}$ is $\nabla \mathbf{x}^{\mathsf{T}}\mathbf{x} = 2\mathbf{x}$. What is the gradient of $\mathbf{x}^{\mathsf{T}}\mathbf{M}\mathbf{x}$, where $\mathbf{M}$ is a symmetric matrix?

(b) Find the $\mathbf{x}$ such that

$$Q(\mathbf{x}) = -\mathbf{b}^{\mathsf{T}}\mathbf{x} + \frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x}$$

is minimized.

(c) Show that the divergence of the three-dimensional vector field $\mathbf{x}$ is $\nabla \cdot \mathbf{x} = 3$.

(d) What is the gradient with respect to $\mathbf{x}$ of

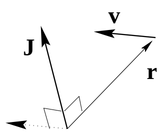
i. $C(\mathbf{x}) = |\mathbf{R} + \mathbf{x}|^2$ ;

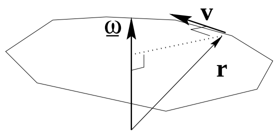
ii. $G(\mathbf{x}) = 1/|\mathbf{R} + \mathbf{x}|$ ?

## Vectors

**M.5** What is $\mathbf{w} = \mathbf{x} \times (\mathbf{y} \times \mathbf{z})$?

**M.6** What is $c = (\mathbf{y} \times \mathbf{z})^2$?




## Rotation and Angular momentum

**M.7** What is the angular momentum, about the origin, of a point mass $m$ with location $\mathbf{r}$ and velocity $\mathbf{v}$?

**M.8** What is the instantaneous velocity $\mathbf{v}$ of a particle at location $\mathbf{r}$ in a rigid body that is rotating with angular velocity $\underline{\omega}$ about an axis passing through the origin? What, in terms of $\underline{\omega}$ and $\mathbf{r}$, is the angular momentum of the particle about the origin? Draw a diagram showing the relationship of the angular momentum vector to $\underline{\omega}$ and $\mathbf{r}$.

The next two questions are new material which you should be able to do using the above results.

**M.9** A rigid body is made up of $N$ point masses $m^{(n)}$ at locations $\mathbf{r}^{(n)}$. What is the angular momentum $\mathbf{J}$ of the rigid body, about the origin, when it rotates about the origin with instantaneous angular velocity $\underline{\omega}$?

Show that $\mathbf{J}$ can be written as a linear function of $\underline{\omega}$,

$$J_i = \sum_k I_{ik}\omega_k, \tag{3}$$

for a certain matrix $\mathbf{I}$, and give an expression for $I_{ik}$.

**M.10** What is the kinetic energy $T$ of the rigid body when its instantaneous angular velocity is $\underline{\omega}$? Show that the kinetic energy can be written as a quadratic form in $\underline{\omega}$, *i.e.*,

$$T = \frac{1}{2}\sum_{ij} \omega_i I_{ij}\omega_j, \tag{4}$$

and give an expression for $I_{ij}$.

# Worked solutions

Solution to exercise M.1 (p. 3):

(a) With experience, matrices like $\mathbf{A}$ can be solved by inspection; but let's use the long route. First, find the eigenvalues, which are the solutions of

$$|\mathbf{A} - \lambda\mathbf{1}| = 0.$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 3 \text{ or } \lambda = 1$$

Second, for each $\lambda$, find a non-zero vector $\mathbf{e}$ such that

$$(\mathbf{A} - \lambda\mathbf{1})\mathbf{e} = 0.$$

The solutions are the two vectors $(1, -1)$ and $(1, 1)$. If we normalize them, we have $\mathbf{e}^{(1)} = (1/\sqrt{2}, -1/\sqrt{2})$ and $\mathbf{e}^{(2)} = (1/\sqrt{2}, 1/\sqrt{2})$.

(b) Using the relationship

$$\mathbf{A} = \sum_a \lambda^{(a)} \mathbf{e}^{(a)} \mathbf{e}^{(a)^{\mathsf{T}}}, \tag{5}$$

we can write

$$\mathbf{A} = 3 \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} + 1 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}. \tag{6}$$

For $\mathbf{x} = (1, 0)$,

$$\mathbf{A}\mathbf{x} = (3/2, -3/2) + (1/2, 1/2) = (2, -1).$$

(c) We express $\mathbf{x}$ as a sum of the eigenvectors by finding its projections onto each of them:

$$x_1 = \mathbf{x} \cdot \mathbf{e}^{(1)} = \frac{1}{\sqrt{2}}; \; x_2 = \mathbf{x} \cdot \mathbf{e}^{(2)} = \frac{1}{\sqrt{2}}; \; \Rightarrow \mathbf{x} = \frac{1}{\sqrt{2}}\mathbf{e}^{(1)} + \frac{1}{\sqrt{2}}\mathbf{e}^{(2)}. \tag{7}$$

(d) When we multiply $\mathbf{S}$ by a vector $\mathbf{e}$, $\mathbf{S}$ interchanges the two components; for $\mathbf{e}$ to be an eigenvector, the two components must therefore be either equal or opposite. The eigenvalues are $\lambda = \pm 1$, with $\mathbf{e} = (1, \pm 1)$. $\mathbf{A}$ and $\mathbf{S}$ commute and their eigenvectors are identical.

(e) *All* vectors are eigenvectors of $\mathbf{1}$ with eigenvalue 1. A convenient pair might be $(1, 0)$ and $(0, 1)$. Another complete set of eigenvectors is $(1, 1)$ and $(1, -1)$.

$\mathbf{A}$ and $\mathbf{1}$ commute, and the eigenvectors of $\mathbf{A}$ are eigenvectors of $\mathbf{1}$; however the converse is not true – not all eigenvectors of $\mathbf{1}$ are eigenvectors of $\mathbf{A}$. [This asymmetry arises because $\mathbf{1}$ is a degenerate matrix.]

**To remember:** If two matrices commute, then it is possible to find a complete set of eigenvectors that are common to both of them. In the above example, $\mathbf{S}$ can be thought of as a symmetry operator. What effect does $\mathbf{S}$ have on a vector $(x_1, x_2)$? It *interchanges* $x_1$ and $x_2$. The fact that $\mathbf{S}$ commutes with $\mathbf{A}$ corresponds to the fact that $\mathbf{A}$ doesn't change if we interchange the indices 1 and 2.

Solution to exercise M.2 (p. 4): Finding the eigenvectors and eigenvalues of a matrix bigger than $2 \times 2$ is, in general, an ugly problem; so if you are asked to find them you can be sure that the matrix must have some special properties you can exploit. Noticing that all three rows of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

are identical, we can write $\mathbf{A}$ as an outer product:

$$\mathbf{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Any matrix like this, $\mathbf{A} = \mathbf{n}\mathbf{n}^{\mathsf{T}}$, has $\mathbf{n}$ as an eigenvector with eigenvalue $\mathbf{n}^2$, and the other eigenvalues are all zero. So here, the eigenvectors are $(1, 1, 1)$ with eigenvalue 3, and any two vectors that are orthogonal to $(1, 1, 1)$, for example $(1, -1, 0)$ and $(1, 1, -2)$, which have eigenvalue zero. [I picked these two by first picking an arbitrary vector orthogonal to $(1, 1, 1)$, namely $(1, -1, 0)$, then taking the cross product of this with $(1, 1, 1)$ to get the third vector.]

Solution to exercise M.3 (p. 4): We solve the equation $\det(\mathbf{S} - \lambda\mathbf{1}) = 0$. The determinant in the case $N = 4$ is

$$|\mathbf{S} - \lambda\mathbf{1}| = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = \lambda^4 - 1, \tag{8}$$

so the eigenvalues are the solutions of

$$\lambda^4 = 1, \tag{9}$$

which are the four fourth-roots of unity, $\lambda = \{1, e^{i\pi/2}, e^{2i\pi/2}, e^{3i\pi/2}\} = \{1, i, -1, -i\}$.

The eigenvectors of the $4 \times 4$ matrix $\mathbf{S}$ are:

$$\begin{array}{ccccc} \lambda & 1 & i & -1 & -i \\ \hline & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} & \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} \end{array} \tag{10}$$

$\mathbf{S}$ is the symmetry operator of a four-fold circularly symmetric system. The above result generalizes to the $N \times N$ matrix corresponding to symmetry under circular rotation through $2\pi/N$. The eigenvalues are the solutions of

$$\lambda^N = 1, \tag{11}$$

and the eigenvectors $\mathbf{f}^{(a)}$ can be written $f_n^{(a)} = e^{i2\pi an/N}$. The transformations to and from the eigenvector basis are the discrete versions of the Fourier transform and the inverse Fourier transform.

# Review of summation convention

| | | |
|---|---|---|
| identity matrix | $\mathbf{1}$ | $\delta_{ij}$ |
| vectors | $\mathbf{a} = \mathbf{b}$ | $a_i = b_i$ |
| inner product | $z = \mathbf{a}^{\mathsf{T}}\mathbf{b}$ | $z = a_i b_i$ |
| outer product | $\mathbf{M} = \mathbf{a}\mathbf{b}^{\mathsf{T}}$ | $M_{ij} = a_i b_j$ |
| matrix multiply | $\mathbf{y} = \mathbf{M}\mathbf{x}$ | $y_i = M_{ij} x_j$ |
| transpose | $\mathbf{z} = \mathbf{M}^{\mathsf{T}}\mathbf{a}$ | $z_i = M_{ji} a_j$ |
| inverse | $\mathbf{M}^{-1}\mathbf{M} = \mathbf{1}$ | $M_{ij}^{-1} M_{jk} = \delta_{ik}$ |
| differentiation | $\mathbf{g} = \nabla f(\mathbf{x})$ | $g_i = \frac{\partial}{\partial x_i} f(\mathbf{x})$ |

The table shows how we can use indices to describe vector and matrix relationships. Wherever an index is repeated in a single term, there is an implicit summation over that index. So $z = a_i b_i$ means $z = \sum_i a_i b_i$. In summation convention, factors within a single term may be written in any order. To translate back into coordinate-free matrix-vector notation, you can get the order right by ensuring that dummy indices appear adjacent to each other. For example, if $V = \sum_{ij} M_{ij} x_i x_j$, then we need to get the $i$s adjacent and the $j$s adjacent: $V = \sum_{ij} x_i M_{ij} x_j$; then we can translate back to $V = \mathbf{x}^{\mathsf{T}}\mathbf{M}\mathbf{x}$. An advantage of index notation over coordinate-free notation is that the meaning of expressions may be more explicit and clear. For example, the inner product $\mathbf{a}^{\mathsf{T}}\mathbf{b}$ is written $a_i b_i$, which is evidently a scalar, not a vector or a matrix, because it has no free indices; whereas the outer product $\mathbf{b}\mathbf{a}^{\mathsf{T}}$, which is not obviously a matrix when written this way, is more clearly seen to be a matrix when we write it in index notation as $b_i a_j$, a term with two free indices, just like a matrix $M_{ij}$.

Solution to exercise M.4 (p. 4):

Choose an index to differentiate with respect to, $j$, which is different from the dummy indices.

(a) $L(\mathbf{x}) = \mathbf{x}^{\mathsf{T}}\mathbf{x} = x_i x_i$

$$\begin{aligned} \frac{\partial}{\partial x_j} x_i x_i &= \frac{\partial x_i}{\partial x_j} x_i + x_i \frac{\partial x_i}{\partial x_j} \\ &= 2\delta_{ij} x_i = 2x_j \end{aligned}$$

So $\nabla \mathbf{x}^{\mathsf{T}}\mathbf{x} = 2\mathbf{x}$.

$$\begin{aligned} \mathbf{x}^{\mathsf{T}}\mathbf{M}\,\mathbf{x} &= x_i M_{ik} x_k \\ \frac{\partial}{\partial x_j}(x_i M_{ik} x_k) &= \frac{\partial x_i}{\partial x_j} M_{ik} x_k + x_i M_{ik} \frac{\partial x_k}{\partial x_j} \end{aligned}$$

$$
\begin{aligned}
&= \delta_{ji} M_{ik} x_k + x_i M_{ik} \delta_{jk} \\
&= (M_{jk} + M_{kj}) x_k \\
&= 2 M_{jk} x_k, \qquad \text{if } M_{jk} \text{ is symmetrical.}
\end{aligned}
$$

So $\nabla \mathbf{x}^{\mathsf{T}} \mathbf{M}\, \mathbf{x} = 2\mathbf{M}\mathbf{x}$.

(b)

$$
\begin{aligned}
Q(\mathbf{x}) &= -\mathbf{b}^{\mathsf{T}}\mathbf{x} + \frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} \\
&= -b_i x_i + \frac{1}{2} x_i A_{ij} x_j \\
\frac{\partial}{\partial x_k} Q(\mathbf{x}) &= -b_i \frac{\partial x_i}{\partial x_k} + \frac{1}{2}\frac{\partial x_i}{\partial x_k} A_{ij} x_j + \frac{1}{2} x_i A_{ij} \frac{\partial x_j}{\partial x_k} \\
&= -b_i \delta_{ik} + \frac{1}{2} \delta_{ik} A_{ij} x_j + \frac{1}{2} x_i A_{ij} \delta_{jk} \\
&= -b_k + \frac{1}{2} A_{kj} x_j + \frac{1}{2} x_i A_{ik}
\end{aligned}
$$

[In standard notation, $\nabla Q = -\mathbf{b} + \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathsf{T}})\mathbf{x}$.] Now find $\mathbf{x}$ such that $\nabla Q = 0$. For simplicity assume $\mathbf{A}$ is symmetric, or else replace $\mathbf{A}$ by $\frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathsf{T}})$; define $\mathbf{A}^{-1}$ by $\mathbf{A}^{-1}\mathbf{A} = \mathbf{1}$, *i.e.*, $A^{-1}_{mn} A_{np} = \delta_{np}$. Now take the statement that the derivative $\nabla_k Q$ is zero,

$$
A_{kj} x_j - b_k = 0,
$$

and left-multiply by $A^{-1}_{nk}$:

$$
\begin{aligned}
\underbrace{A^{-1}_{nk} A_{kj}}\, x_j &= A^{-1}_{nk} b_k \\
\delta_{nj} \quad x_j &= A^{-1}_{nk} b_k \\
\Rightarrow x_n &= A^{-1}_{nk} b_k; \\
\text{or, in standard notation, } \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}.
\end{aligned}
$$

(c)

$$
\begin{aligned}
\nabla \cdot \mathbf{x} &= \frac{\partial}{\partial x_i} x_i \\
&= \delta_{ii} \\
&= \sum_i 1 = N,
\end{aligned}
$$

where $N = 3$, the dimension of the unit matrix $\delta_{ij}$.

(d) i.

$$
\begin{aligned}
C(\mathbf{x}) &= (R_i + x_i)(R_i + x_i) \\
&= (R_i + x_i)^2 \quad \text{This } i \text{ is still a dummy index.} \\
\frac{\partial C}{\partial x_j} &= 2(R_i + x_i) \frac{\partial}{\partial x_j}(R_i + x_i) \\
&= 2(R_i + x_i)(0 + \delta_{ij}) \\
&= 2(R_j + x_j) \\
\textit{i.e.}, \ \nabla C &= 2(\mathbf{R} + \mathbf{x})
\end{aligned}
$$

ii.

$$
\begin{aligned}
G(\mathbf{x}) &= \frac{1}{\sqrt{(R_i+x_i)^2}} \\
\frac{\partial G}{\partial x_j} &= -\frac{1}{2}\frac{1}{\left[(R_i+x_i)^2\right]^{3/2}}\frac{\partial}{\partial x_j}\left[(R_i+x_i)^2\right] \\
&= \frac{-2(R_j+x_j)}{2\left[(R_i+x_i)^2\right]^{3/2}} \\
\textit{i.e.},\ \nabla G &= -\frac{(\mathbf{R}+\mathbf{x})}{|\mathbf{R}+\mathbf{x}|^3}.
\end{aligned}
$$

## Cross products

*In three dimensions,* we can define a cross product of two vectors. In order to describe cross products, it is useful to introduce the permutation symbol $\epsilon_{ijk}$ which is defined as follows:

$$
\begin{aligned}
\epsilon_{123} = \epsilon_{312} = \epsilon_{231} &= 1 \\
\epsilon_{213} = \epsilon_{321} = \epsilon_{132} &= -1 \\
\epsilon_{ijk}\ \text{(for any other values of } i,j,k) &= 0.
\end{aligned}
$$

This allows us to rewrite the vector equation

$$\mathbf{a} = \mathbf{b}\times\mathbf{c}, \tag{12}$$

normally written as

$$
\begin{array}{rcl}
a_1 &=& b_2c_3 - b_3c_2 \\
a_2 &=& b_3c_1 - b_1c_3\ , \\
a_3 &=& b_1c_2 - b_2c_1
\end{array} \tag{13}
$$

in the briefer form:

$$a_i = \epsilon_{ijk}\, b_j\, c_k. \tag{14}$$

If we memorise the identities

$$\sum_i \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{15}$$

$$\text{and}\quad \epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} \tag{16}$$

then we can reproduce any equation involving cross products.

### Examples

Solution to exercise M.5 (p. 4): What is $\mathbf{w} = \mathbf{x}\times(\mathbf{y}\times\mathbf{z})$?

$$
\begin{aligned}
w_i &= \epsilon_{ijk}x_j\epsilon_{klm}y_lz_m & (17)\\
&= \epsilon_{kij}\epsilon_{klm}x_jy_lz_m & (18)\\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})x_jy_lz_m & (19)\\
&= x_jy_iz_j - x_jy_jz_i & (20)\\
&= x_jz_jy_i - x_jy_jz_i, & (21)
\end{aligned}
$$

*i.e.*,

$$\mathbf{w} = \mathbf{x}\times(\mathbf{y}\times\mathbf{z}) = (\mathbf{x}\cdot\mathbf{z})\mathbf{y} - (\mathbf{x}\cdot\mathbf{y})\mathbf{z}. \tag{22}$$

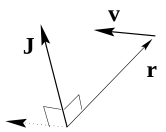
[Here, $\mathbf{x}\cdot\mathbf{z}$ is another way of writing $\mathbf{x}^{\mathsf{T}}\mathbf{z}$.]

Solution to exercise M.6 (p. 4): What is $c = (\mathbf{y} \times \mathbf{z})^2$?

$$c = \epsilon_{ijk} y_j z_k \epsilon_{ilm} y_l z_m \tag{23}$$
$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) y_j z_k y_l z_m \tag{24}$$
$$= y_j z_k y_j z_k - y_j z_k y_k z_j \tag{25}$$
$$= y_j y_j z_k z_k - y_j z_j y_k z_k \tag{26}$$

*i.e.*,

$$c = (\mathbf{y} \times \mathbf{z})^2 = (\mathbf{y}^\mathsf{T}\mathbf{y})(\mathbf{z}^\mathsf{T}\mathbf{z}) - (\mathbf{y}^\mathsf{T}\mathbf{z})^2 \tag{27}$$
$$= \mathbf{y}^2\mathbf{z}^2 - (\mathbf{y}^\mathsf{T}\mathbf{z})^2. \tag{28}$$




Solution to exercise M.7 (p. 5): By definition,

$$\mathbf{J} = m\,\mathbf{r} \times \mathbf{v}, \tag{29}$$

or

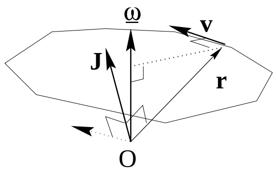
$$J_i = m\,\epsilon_{ijk}\,r_j\,v_k. \tag{30}$$

Solution to exercise M.8 (p. 5):

$$\mathbf{v} = \underline{\omega} \times \mathbf{r}, \tag{31}$$

or

$$v_i = \epsilon_{ijk}\,\omega_j\,r_k. \tag{32}$$




The angular momentum is

$$\mathbf{J} = m\,\mathbf{r} \times \mathbf{v} = m\,\mathbf{r} \times (\underline{\omega} \times \mathbf{r}). \tag{33}$$

$\mathbf{J}$ is perpendicular to $\mathbf{r}$ and $\mathbf{v}$, so in general it is *not* parallel to $\underline{\omega}$.

Solution to exercise M.9 (p. 5): The angular momentum $\mathbf{J}^{\mathrm{TOT}}$ is the sum of the individual angular momenta (29) for each particle, with the velocity of each particle being given by (31). We expect this to yield a vector that is a linear function of $\underline{\omega}$, $J_i = I_{ik}\omega_k$, for some matrix $\mathbf{I}$.

$$J_i^{\mathrm{TOT}} = \sum_n m^{(n)} \epsilon_{ijk} r_j^{(n)} v_k^{(n)} \tag{34}$$
$$= \sum_n m^{(n)} \epsilon_{ijk} r_j^{(n)} \epsilon_{klm} \omega_l r_m^{(n)} \tag{35}$$
$$= \sum_n m^{(n)} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) r_j^{(n)} \omega_l r_m^{(n)} \tag{36}$$
$$= \sum_n m^{(n)} (r_j^{(n)} \omega_i r_j^{(n)} - r_j^{(n)} \omega_j r_i^{(n)}) \tag{37}$$
$$= \sum_n m^{(n)} (r_j^{(n)} \omega_i r_j^{(n)} - r_j^{(n)} \omega_j r_i^{(n)}) \tag{38}$$
$$= \left[\sum_n m^{(n)} (r_j^{(n)} r_j^{(n)} \delta_{ik} - r_i^{(n)} r_k^{(n)})\right] \omega_k \tag{39}$$

Thus the moment of inertia tensor is

$$I_{ik} \equiv \left[\sum_n m^{(n)} (r_j^{(n)} r_j^{(n)} \delta_{ik} - r_i^{(n)} r_k^{(n)})\right] \tag{40}$$

If we happen to be in the basis in which $\mathbf{I}$ is a diagonal matrix – there must be such a basis, since $\mathbf{I}$ is a real symmetric matrix – then $\mathbf{I}$ can be written longhand as follows (with the suffices $n$ omitted from the $r$s):

$$\mathbf{I} = \sum_n m^{(n)} \begin{bmatrix} (r_2^2 + r_3^2) & 0 & 0 \\ 0 & (r_1^2 + r_3^2) & 0 \\ 0 & 0 & (r_1^2 + r_2^2) \end{bmatrix}. \tag{41}$$

Solution to exercise M.10 (p. 5): What is the kinetic energy $T$ of the rigid body when its instantaneous angular velocity is $\underline{\omega}$? We expect the kinetic energy to be a quadratic form in $\underline{\omega}$, *i.e.*, $T = \frac{1}{2}\sum_{ij} \omega_i I_{ij} \omega_j$.

$$\begin{aligned} T &= \frac{1}{2}\sum_n m^{(n)}[\mathbf{v}^{(n)}]^2 && (42) \\ &= \frac{1}{2}\sum_n m^{(n)} v_i^{(n)} v_i^{(n)} && (43) \\ &= \frac{1}{2}\sum_n m^{(n)} \epsilon_{ijk}\, \omega_j\, r_k^{(n)} \epsilon_{ilm}\, \omega_l\, r_m^{(n)} && (44) \\ &= \frac{1}{2}\sum_n m^{(n)} (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \omega_j r_k^{(n)} \omega_l r_m^{(n)} \\ &= \frac{1}{2}\sum_n m^{(n)} \left( \omega_j r_k^{(n)} \omega_j r_k^{(n)} - \omega_j r_k^{(n)} \omega_k r_j^{(n)} \right) \\ &= \frac{1}{2}\omega_j \left[ \sum_n m^{(n)} \left( r_i^{(n)} r_i^{(n)} \delta_{jk} - r_k^{(n)} r_j^{(n)} \right) \right] \omega_k \\ &= \frac{1}{2}\omega_j I_{jk} \omega_k, && (45) \end{aligned}$$

where $\mathbf{I}$ has the same definition as before.

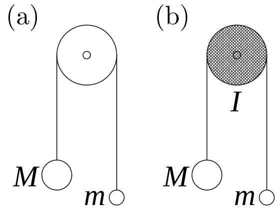
In coordinate-free notation,

$$\mathbf{I} = \left[ \sum_n m^{(n)} \left( \mathbf{r}^{\mathsf{T}} \mathbf{r} \right) \mathbf{1} - \mathbf{r}\mathbf{r}^{\mathsf{T}} \right], \tag{46}$$

where again we have omitted the suffices '$n$' from the $\mathbf{r}$ vectors to avoid clutter.

# 2 Traditional Problems

## Energy method; dimensional analysis




**T.1** Pulleys.

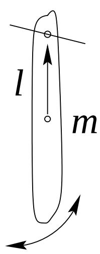
(a) Two masses $M$ and $m$ are suspended over a massless pulley. Using the energy method, find the acceleration of mass $M$. Also solve the problem by finding the forces acting.

(b) Two masses $M$ and $m$ are suspended over a pulley of radius $a$ whose moment of inertia is $I$. Find the acceleration of mass $M$, assuming the rope does not slip.

(c) What can dimensional analysis alone say about these two problems?

**T.2** Spring 1. A mass $m$ is suspended from an ideal spring of constant $k$ and unstretched length $l$. The mass is free to move vertically. Use the energy method to find the equation of motion. What happens to the frequency of small oscillations if the system is put on the moon, where the strength of gravity is six times smaller than on the earth?

How much of this question can be answered using dimensional analysis?




**T.3** Compound pendulum. A compound pendulum is a rigid body that pivots about a horizontal axis. The pendulum has mass $m$, and the centre of mass is a distance $l$ from the pivot. The moment of inertia about the centre of mass is $I_0$. Use the energy method to find the period of small oscillations of this pendulum. Sketch the period as a function of $l$, the distance of the pivot from the centre of mass. Extend your plot to include negative values of $l$ as follows: assume a negative value of $l$ denotes a pivot point a distance $|l|$ from the centre of mass, but on the opposite side from that shown in the figure.

**T.4** Safety rope. You cut off a length $l$ of stretchable rope from a reel. [$l$ is the unstretched length.] Use a thought experiment to deduce how the spring constant $k$ of the piece of rope depends on the chosen length $l$. [Ans: $k = k^*/l$, where $k^*$ is a property of the rope that is independent of length.] A rock climber, with weight $mg$, climbing a vertical cliff, attaches a length $l$ of stretchable rope between himself and an adjacent piton, then immediately falls off. Use dimensional analysis to predict how the maximum force $F$ exerted by the rope on the climber depends on $l$.

Solve for $F$ using Newtonian methods, making approximations appropriate for the cases where (a) the rope is very stiff (*i.e.*, has large $k^*$) (b) the rope is very stretchable (*i.e.*, has small $k^*$).

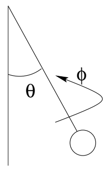
**T.5** Oscillation. A particle of mass $m$ moves in one dimension in the potential

$$V(x) = -\frac{A}{x} + \frac{B}{x^{12}}.$$

Sketch this potential. Find the equilibrium point, $x_0$, and show that, for small displacements, the system performs simple harmonic motion around this point. Find the frequency of the simple harmonic motion. [Hint: Taylor-expand $V(x)$ to quadratic order around $x_0$.]

For the case $A = 1$, $B = 1$, make a graph of $V(x)$ and the Taylor expansion of $V(x)$ around $x_0$, for $x$ from 0 to $2x_0$. (Restrict the vertical range to $\pm 1$.) Make a second, more detailed, graph near the minimum.

**T.6** Conical pendulum. A point mass $m$ on the end of a light string of length $l$ is free to swing as a conical pendulum. Show that, in terms of the (constant) angular momentum $J$ of the mass about the vertical axis, the energy of the pendulum may be written as




$$E = V_{\text{eff}}(\theta) + \frac{1}{2} m l^2 \dot{\theta}^2,$$

where

$$V_{\text{eff}}(\theta) = mgl(1 - \cos\theta) + \frac{J^2}{2ml^2 \sin^2\theta}$$

is the effective potential that determines the motion in $\theta$. Sketch the effective potential. By differentiating $V_{\text{eff}}$ twice with respect to $\theta$, show

(a) that the mass can move steadily round a circle, with $\theta = \theta_0$ and angular velocity $\Omega$ given by

$$\Omega^2 = \frac{g}{l\cos\theta_0},$$

(b) that, if the pendulum is then given a little extra energy without changing its angular momentum, $\theta$ oscillates about $\theta_0$ with angular frequency $\omega$ given by

$$\omega^2 = \Omega^2(1 + 3\cos^2\theta_0).$$

[Hint: see question T.5]

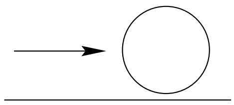
Use these results to discuss the precession of almost-circular orbits of a conical pendulum, assuming $\theta_0 \ll 1$.

## Collisions

The Topic of Collisions is one that you have already studied, and you have all the tools you need to solve collision problems: (a) you can usually identify a momentum and an angular momentum that are conserved during a collision; and (b) energy is conserved between collisions, though not necessarily during a collision. Remember that you can consider angular momentum about any convenient point; the point of impact is often a handy choice, since the impulse exerts no couple about this point. We will not spend any lecture time on collisions, since there is nothing new to say, but the difficulty of the problems you may encounter has increased. In every problem you should draw a free body diagram showing all the forces acting on each body during the collision.

Friction is another topic that will not be lectured, but you should be able to use the following standard model of friction. The dynamic coefficient of friction $\mu$ gives the frictional force $\mathbf{f}$ in terms of the perpendicular force $\mathbf{w}$ between the two surfaces: $|\mathbf{f}| = \mu|\mathbf{w}|$; the direction of the frictional force is opposite to the direction of the relative motion between the surfaces. Note that the magnitude of the frictional force is independent of the relative velocity.

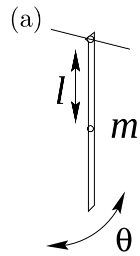
**T.7** Door stop. A door swings open and hits a wall. Where should a rubber doorstop be placed on the wall to minimize the force acting on the *hinges* of the door during the collision?

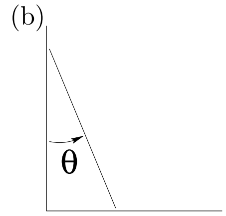


**T.8** Snooker. A stationary snooker ball of radius $a$ receives a horizontal impulse along a line passing through the centre of the ball. Assuming the coefficient of friction between ball and table is $\mu$, sketch a graph of the velocity of the ball $v$ and its angular velocity $\omega$ as a function of time after the impulse. (Sketch $a\omega$ on the same graph as $v$.) [The moment of inertia of a ball about its centre of mass is $I_0 = \frac{2}{5}ma^2$.]

[Optional extra: Assuming the impulse delivered to the ball is horizontal, where should the snooker ball be hit in order for it to immediately roll without slipping?]

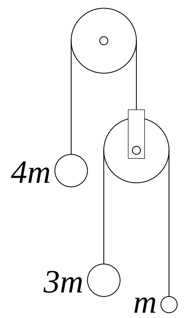
## Lagrangian and Hamiltonian dynamics




**T.9** Ladder.(a) Find the Lagrangian of a compound pendulum made from a thin rod of length $2l$ suspended from one end, and find its equation of motion using Lagrangian methods (c.f. question T.3).

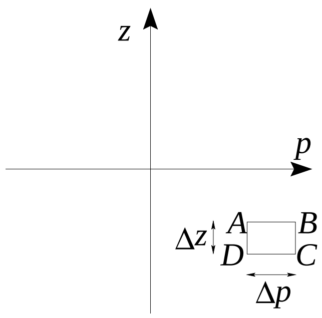
(b) A ladder of length $2l$ stands on a frictionless floor and leans against a frictionless wall. Assuming it remains in contact with both of them, evaluate the Lagrangian and the conjugate momentum and find the equation of motion.




**T.10** Pulley Galore. (a) Redo T.1(a) using Lagrangian methods.

(b) A light string hanging over a massless pulley carries a mass $4m$ on one end and a second pulley on the other end, over which another string carries masses $3m$ and $m$ as shown. Using a suitable pair of generalized coordinates, write down the Lagrangian function for the system and the Euler–Lagrange equations. Deduce the accelerations of the three masses, assuming they start at rest.

Why does the first pulley turn? The masses on the two sides balance!




**T.11** Vertical state space. A point mass moves vertically in a uniform gravitational field. Write down the Lagrangian $L(z, \dot{z})$, and derive the conjugate momentum $p$ and the Hamiltonian. [Express the Hamiltonian as a function of $z$ and $p$.] Write down Hamilton's equations.

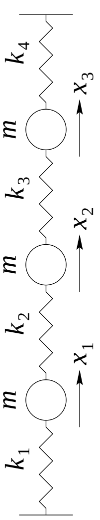
Sketch a trajectory of the system in the state space $z, p$. Show on the state space diagram the states $A', B', C', D'$ reached from the four starting points $A, B, C, D$ shown, after a duration $t$. [Pick any convenient time $t$ and make clear in your diagram the relationship between the points $A', B', C', D'$.] Show that the area in state space of the polygon $A'B'C'D'$ is equal to the area of $ABCD$.

This very general result is called Liouville's theorem: *in a Hamiltonian system, state space volume is conserved.*

**T.12** Conical pendulum II.

Rework the conical pendulum problem T.6 using Lagrangian and Hamiltonian methods as follows. Find the Euler–Lagrange equations. Also find the Hamiltonian, expressing it as a function of the state $(\theta, \phi)$ and the conjugate momenta $(p_\theta, p_\phi)$, and write down Hamilton's equations. [Optional: From Hamilton's equations, find the condition on $\theta$ for $\dot{p}_\theta = 0$, and the frequency of small oscillations of $\theta$ about this value.]

## Matrices

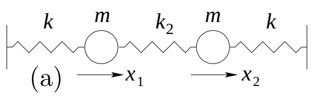



Reminder: you should be familiar with 1A Mathematics material on matrices, as reviewed by the exercises and worked solutions starting on page 3.

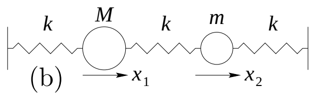
**T.13** Displaced springs. Three masses are connected by four springs as shown. The displacements from equilibrium of the three masses are $x_1, x_2, x_3$. Evaluate the three forces $f_1, f_2, f_3$ acting on the three masses and show that they can be written in the form $\mathbf{f} = -\mathbf{K}\mathbf{x}$, where $\mathbf{K}$ is a $3 \times 3$ matrix. Evaluate the potential energy as a function of $\mathbf{x}$, and show that it can be written as a quadratic form.

Now, for simplicity, assume that $k_1 = k_2 = k_3 = k_4$. For $i = 1, 2, 3$, sketch the displacements of all three masses when a unit force is applied at point $i$, assuming the system is at equilibrium. Show that, for any $j$ and $i$, the displacement at $j$ when a unit force is applied at $i$ is equal to the displacement at $i$ when a unit force is applied at $j$. [This property is very general.]

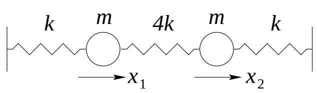
## Normal modes




**T.14** Two masses. (a) Find the frequencies of the normal modes of the two-mass system shown, and sketch how they vary with $k_2$, the spring constant of the centre spring. Describe how the normal modes vary with $k_2$.



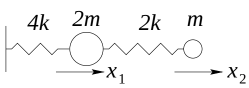

(b) Find the frequencies of the normal modes of the two-mass system shown, and sketch how they vary with $M$ (show the ratio $M/m$ varying from 1 to $\infty$). Annotate your sketch with pictures showing the normal modes associated with each eigenvalue at $M \simeq m$ and for $M \gg m$.




**T.15** Two mass II. (a) Find the normal modes of the two-mass system shown, and their frequencies.

(b) Starting from the equilibrium position, the right-hand mass is displaced a unit distance while the left-hand mass is held still. Both masses are released at $t = 0$. Find and sketch the subsequent motion of the two masses as a function of time. Also sketch the amplitudes of the two normal modes. [Hint: the general motion of the system is $\mathbf{x}(t) = \sum_a A_a \cos(\omega_a t + \phi_a)\mathbf{e}^{(a)}$.]

(c) Starting from the equilibrium position, the right-hand mass is given a *kick* to the right (a unit impulse). Find the amplitudes of the two normal modes and the motion of the two masses. Contrast with the answer to (b).

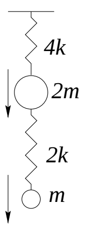



**T.16** Two mass III. Two masses are connected to a wall by springs as shown. The system is placed on a frictionless horizontal surface.

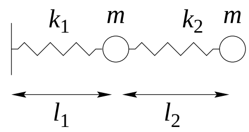
(a) Find the normal modes and their frequencies. [Hint: solve the generalized eigenvector problem $[\mathbf{K} - \omega^2\mathbf{M}]\mathbf{e} = 0$.] Sketch the normal modes.

(b) Starting from the equilibrium position, the right-hand mass is displaced a unit distance while the left-hand mass is held still. Both masses are released at $t = 0$. Find and sketch the subsequent motion of the two masses as a function of time. Also sketch the amplitudes of the two normal modes.

(c) The system is now suspended from the end that used to be attached to the wall. What are the normal modes of vertical motion?

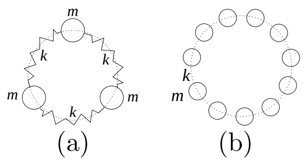


**T.17** Divided spring. Two masses of mass $m$ are attached to an ideal uniform light spring. The first is attached a distance $l_1$ from one fixed end, and the second is attached a distance $l_2$ further along at the far end. [These are unstretched lengths.] Let the spring constants of the two sub-springs be $k_1$ and $k_2$. [Think: is $k_1/k_2 = l_1/l_2$, or $l_2/l_1$?] The system is placed on a frictionless horizontal surface. The normal modes of the system are found (for horizontal motion along the line of the springs). Given that the lowest frequency normal mode has the displacement vector $(1, 2)$, *i.e.*, the second mass moves twice as far as the first one, deduce the ratios $k_1 : k_2$ and $l_1 : l_2$.

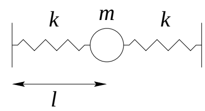



Find the higher frequency normal mode and the ratio of the frequencies of the two modes.

[Hint for the first part: write down the two net forces when the displacement is $(1, 2)$; what must the ratio of these two forces be? Hint for the second part: what do you know about eigenvectors of symmetric matrices? Ans: $3 : 2$; $\sqrt{6} : 1$.]

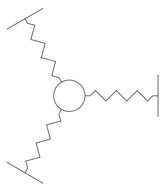



**T.18** Symmetries. (a) Three masses connected by identical springs are constrained to move on a circle. What are the normal modes? (b) Eleven masses connected by identical springs are constrained to move on a circle. What are the normal modes?



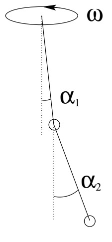

**T.19** 3D spring. (a) Two ideal springs with unstretched length $l_0$ and spring constant $k$ are stretched to length $l$ and attached to a point mass $m$ as shown. The mass is free to move in all three dimensions. Describe all the normal modes of this system and find their frequencies.

What happens to the normal modes if $l < l_0$?



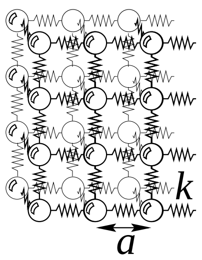
(b) *Without calculation,* describe the normal modes of a similar system in which three identical springs are arranged symmetrically around the mass. [You may find it helpful to consider the case with four springs surrounding the mass first.] Describe the potential $V(\mathbf{x})$ to quadratic order for displacements $\mathbf{x}$ lying in the plane. Describe the motion if the mass is given a small kick from the centre in an arbitrary direction in the plane.

**T.20** Driven system. The two-mass system of question T.14(b), with $M = m$, is driven by a force $f \sin(\omega t)$ applied to the first mass, where $\omega$ is not equal to either of the normal mode frequencies. As with a simple harmonic oscillator, the response can be represented as the sum of a steady-state response of frequency $\omega$ and a free response. Find the steady-state response of the system. [Hint: see maths recap question M.1(c) (p. 3).] Sketch the amplitude of the responses of $x_1$ and $x_2$ as a function of $\omega^2$.



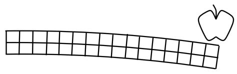

**T.21** Double pendulum. A planar double pendulum consisting of two masses and two light rods is spun about a vertical axis passing through its suspension point at fixed angular velocity $\omega$. Find the Lagrangian and the equation of motion, assuming the state of the pendulum is close to the vertical $(\alpha_1, \alpha_2) = (0, 0)$. Find the normal modes of the system, and their frequencies (a) for $\omega = 0$; (b) for general $\omega$. Find the critical $\omega$ above which the vertical state $(\alpha_1, \alpha_2) = (0, 0)$ is not stable, and describe what happens if $\omega$ exceeds this value.

## Elasticity




**T.22** Model steel. Consider a crude model of steel: a collection of particles connected by springs. Estimate the spacing $a$ and the spring constant $k$ using your knowledge of properties of matter. [Crude estimates are fine. Example of estimating $k$ given $a = 3 \times 10^{-10}$ m: model the interatomic potential by a quadratic function with minimum at spacing $a$, and depth 5eV, and with curvature such that the potential is zero when the displacement is $a/2$, gives $k \simeq 4\,\mathrm{eV}/10^{-20}\,\mathrm{m}^2 = 64\,\mathrm{N/m}$].

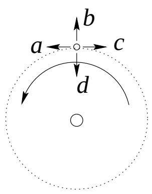
Relate $k$ and $a$ to the Young's modulus, and deduce the Young's modulus of this model steel.



Now estimate the deflection of a steel 30 cm ruler, 1 mm thick, when an apple (weight = 1 Newton) is placed on its free end, the other end being clamped horizontal. [The ruler is shown in side view in the figure.] Assume that the upper half of the ruler is stretched and the lower half is compressed. How does the answer depend on the thickness of the ruler?

**T.23** Write a short essay explaining the equivalence of a shear strain to superposed compression and extension.

## Orbits




**T.24** Ellipses. Sketch the four orbits resulting when a satellite in a circular orbit is given a small impulse in each of the four directions shown. In each case, state the changes in energy, angular momentum, and period of the satellite.

**T.25** Power law potentials. Sketch the forms of effective radial potential for a power-law central force $F = -Ar^n$ with (a) $n = 1$; (b) $n = -1$; (c) $n = -6$. Describe the types of motion that are possible in each case.

In case (a), describe the motion in terms of $r, \theta$, finding the condition for a circular orbit, and the frequency of the radial oscillations resulting from a radial perturbation; then find the potential as a function of $(x, y) = (r\cos\theta, r\sin\theta)$, and describe the motion in terms of $x$ and $y$. What is the relationship of case (a) to the conical pendulum (T.6)?

## Rotating frames

**T.26** Vertical. (a) Estimate the angular difference between the vertical and the line towards the centre of the earth in Cambridge.

(b) A stone is dropped from a stationary helicopter $h = 500$ m above the ground at the equator. How far from the point vertically beneath the helicopter does it land and in what direction? You should try to solve this problem in two ways: (i) by considering the angular momentum of the stone (harder), and (ii) by Coriolis force (easier). Always make a crude estimate first. [Ans: 24 cm to the East]

**T.27** Missile. A cannonball is fired from Oxford in the direction of Cambridge. Assuming a nearly-horizontal muzzle velocity of magnitude $v$, and assuming they get the range right, how far away from Cambridge does the missile land on account of the Coriolis force? [Try $v = 1000\,\mathrm{ms}^{-1}$.] Discuss how your answer depends on the velocity. [Neglect air resistance.]

**T.28** Circular coordinates. (a) A point mass moves in a two-dimensional plane in a potential $V(r, \theta)$. Describing the motion using cylindrical coordinates $(r, \theta)$, find the Lagrangian, and the Euler–Lagrange equations for $r$ and $\theta$.

(b) Now assume that the coordinates $(r, \theta)$ are defined relative to a frame rotating about the origin at angular velocity $\omega$. Write down the Lagrangian and find the Euler–Lagrange equation for the coordinate $r$. Compare the answer with the equation of motion for the inertial frame of part (a).

## Rigid bodies

**T.29** Disc. A uniform disc of radius 0.1 m and mass 0.4 kg is rotating with angular velocity 1 rad $s^{-1}$ about an axis at 45° to its plane through its centre of mass. What is (a) its angular momentum, and (b) its kinetic energy? [Assume the centre of mass is stationary. Ans: $(0.7, 0, 1.4) \times 10^{-3}$ kg $m^2$ $s^{-1}$ w.r.t. obvious axes; $\frac{3}{4}$ mJ]

**T.30** Tile. A uniform rectangular tile of mass $M$ drops without spinning until its corners reach positions $(0, 0, 0)$, $(2a, 0, 0)$, $(2a, 2b, 0)$, $(0, 2b, 0)$, when it strikes the top of a vertical pole at a point very close to the $(0, 0, 0)$ corner. Just before impact the velocity of the tile was $(0, 0, -u)$. Assuming that the tile does not break, and that the impact is elastic (i.e. the kinetic energy of the tile is conserved), find immediately after impact

(a) the velocity of its centre;

(b) the angular momentum about its centre;

(c) its angular velocity;

(d) the velocity of the corner at $(0, 0, 0)$.

[Ans: $\frac{5}{7}(0, 0, -u)$, $\frac{2}{7}Mu(-b, a, 0)$, $\frac{6}{7}u(-\frac{1}{b}, \frac{1}{a}, 0)$, $(0, 0, +u)$]

**T.31** Precession of earth. Estimate how big the equatorial bulge of the earth is. [Ans: the bulge is about $R_e/300$ in thickness.]

*Estimate* how fast the earth precesses because of the gravitational torque arising from its bulge (pretend that the bulge is concentrated in a ring round the equator), and express your answer in units of zodiacal signs per millenium. (When the earth's axis precesses through 180 degrees, the change is six signs of the zodiac.)

[For background information see `wol.ra.phy.cam.ac.uk/teaching/dynamics/`. Ans: about one zodiacal sign every two millenia – which explains why zodiacal signs and birthdays no longer match!]

## 3 Quickies

**Q.1** Kettle. How high would a kettle-full of water go if you put the energy required to boil the water into translational kinetic energy of the water instead? [What about a cup-full?]

**Q.2** Liquid length. The latent heat of vaporization of a liquid, per unit volume, is an energy per length$^3$. The surface tension of a liquid is an energy per length$^2$. The ratio of these is a length, which is an intrinsic property of the liquid. What length does it correspond to? Work out its value for water. [If you don't have the surface tension and latent heat of water to hand, estimate them.]

**Q.3** 50p. At what frequency does a vertical 50p piece oscillate when it rolls to and fro on one edge? Assume the edge is the arc of a circle whose centre is the opposite vertex.

**Q.4** General relativity. Use dimensional analysis to predict the deflection of starlight passing near the sun, as a function of the distance $d$ of the ray from the centre of the sun.

**Q.5** Cup. The ringing note produced by a tea cup when it is tapped on the rim with a spoon is liable to vary in pitch and purity depending on whereabouts in relation to the handle the cup is tapped. Predict this variation.

**Q.6** Corrugations. A sheet of thickness 1 mm is corrugated to a depth of 50 mm. **Estimate** the factor by which its stiffness to bending about an axis perpendicular to the corrugations exceeds that of the uncorrugated sheet. [Ans: about 2500]

**Q.7** Train. How fast would a train travelling along a straight track have to go to risk being tipped over by the Coriolis force? [And how fast must a train go to go into orbit?]

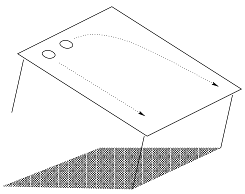
**Q.8** Bath. Discuss the claim that the water corkscrews out of the bath in opposite directions in the north and south hemispheres. Compare the magnitude of the effect (at the north pole) with the typical angular momentum introduced by a foot testing the water in the bath. [A friend met a creative tourist-pleaser in Ecuador who demonstrated the rotating flow from a pair of bathtubs, one each side of the equator line!]

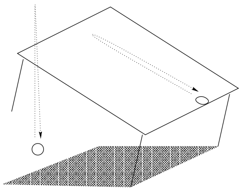
**Q.9** Atmosphere. Assuming the vertical velocity of a nitrogen molecule is given by $mv_z^2/2 = kT/2$, what height does it go to, assuming no collisions?

What is the pressure exerted by a column of air (density 1 kg/m$^3$) of that height?

# 4 Deep thought

**D.1** Conveyor belt. You are standing on an airport conveyor belt moving at $1\,\mathrm{ms}^{-1}$ with a trolley; you give yourself a push and hop on the trolley so that you are rolling along at $1\,\mathrm{ms}^{-1}$ relative to the conveyor, in the same direction. How much work have you done? What is your change in kinetic energy, from the point of view of an observer stationary on the ground? Can we solve the world's energy problems?

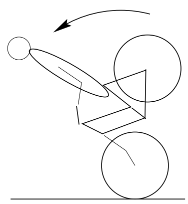




**D.2** Shove ha'penny. (a) Two pennies are released simultaneously from the same height on a sloping table. The right-hand penny is given a horizontal kick at exactly the moment of release. Both pennies slide and experience sliding friction. Which penny falls off the table first? [Think about it, then try the experiment!] [The standard model of sliding friction asserts that the magnitude of the friction force is proportional to the perpendicular force and independent of the sliding speed.] [The experimental results are clearest if the table's angle is chosen so that the friction is almost big enough to stop the coin from sliding.]

(b) A penny is launched up a sloping table, goes up, and comes down. Does it take more time to come down than go up, or less time, or just the same? Answer the same question for a light ball thrown straight up in the air. (Assume there *is* air resistance.)

**D.3** European union. If everyone in the UK started driving on the right instead of the left, how big would be the change in rotation rate of the earth?



**D.4** Locked brakes. A moving cyclist jams on her front brakes. Estimate how fast she needs to be going in order to roll right over, assuming that the front wheel does not skid.

What happens when the *rear* brakes only are jammed on? [Where does the angular momentum in the rear wheel go? Does the back wheel skid?] At what speed is there a danger of hurtling over the front wheel?

**D.5** Walking. What is the fastest walking speed of a two-legged animal with legs of length $l$? What is the pacing frequency of the animal, if it walks at that speed? Do big animals pace slower or faster? Does this fit with your observations of crows and giraffes (which walk in the style of two-legged animals)?

**D.6** Bouncing balls. A steel ball bounces on a hard floor. Draw the forces acting on the ball at mid-bounce. What is the typical force on the ball during the bounce? Estimate the contact time. (For simplicity, model the ball by a cube or a cylindrical rod.) Where does the kinetic energy go, at the bottom of the bounce? If the collision is 'elastic', how does all the energy manage to get back into kinetic energy of the ball? Where is the energy stored if the floor is made of harder material than the ball, and where if it is softer? Is perfect elasticity possible?

It may help to consider the special case of a long compressible cylinder hitting a hard floor.

**D.7** Burn time. A satellite is in an elliptical orbit about a planet and is short of fuel.

When is the best time in the orbit for the thrusters to be fired if the aim is for the satellite to leave the planet, and in what direction should they be fired?

**D.8** Space elevator. One suggestion for putting satellites into orbit cheaply without using rockets is to build a tower on the equator 40,000 km high containing an elevator. One would put the payload in the elevator, lift it to the top, and just step out into orbit. Estimate the strain at the foot of the tower if it is made of steel. (Steel is about 8 times as dense as water, which has a density of $1000\,\mathrm{kg/m^3}$.) Do you think the space elevator is a sensible suggestion? To reduce the pressure on the ground, could the tower be made higher still, so that the 'centrifugal force' on its extreme parts holds the tower up?

**D.9** Cornering. A cyclist is freewheeling on a well-oiled bicycle on a flat plane. Starting out moving upright in a straight line, she then steers so as to go in a circle, with the bicycle leaning over in the usual way. Her centre of mass remains at the same distance from the ground contact. Since her centre of mass is lower when leaning, the gravitational potential energy decreases, so by energy conservation the bike speeds up. Where did the force come from to make her go faster?

**D.10** Bad working. Find all the mistakes in the following piece of reasoning.

> The gravitational potential per unit mass outside a sphere of mass $m$ is
> $$V(r) = Gm/r.$$
> If the earth is uniform, then, as we go towards its centre, we can ignore the shell outside our radius, so the effective attracting mass is $m \propto \rho r^3$. Thus, at a point inside the earth, the potential is
> $$V(r) \propto \rho G r^2,$$
> and the force is
> $$-dV/dr \propto -\rho G r.$$
> This restoring force is proportional to the distance from the centre, so a particle falling through the centre of the earth would perform simple harmonic motion.

Do the calculation correctly.

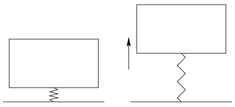
**D.11** Tides. Why are there two high tides per day?

Why are the tides dominated by the moon, and not by the sun, which exerts a bigger gravitational force?

Estimate the height of the mid-ocean tide.

Why is there sometimes only one high tide per day in the North Pacific Ocean?

[Mass of moon $\simeq 0.01\times$ mass of earth.] [Ans: Tidal range in Honolulu is of order 1 foot.]



**D.12** Zebedee. A mass sits atop a strong light uniform spring, held compressed by a latch. The latch is released, and the mass is launched into the air. The spring is not attached to the ground or the mass. How high does the spring go, compared with the mass?

**D.13** Car areas. The efficiency of a car is sometimes expressed in miles per gallon. What are the dimensions of this quantity? How big is it? What is its interpretation?

**D.14** Anharmonic potentials that are isochronous. The harmonic potential $V(x) = kx^2/2$ has the special property that the period of oscillations under the equation of motion $md^2x/dt^2 = -dV/dx$ is the same for all amplitudes. Do any other potentials have this property?

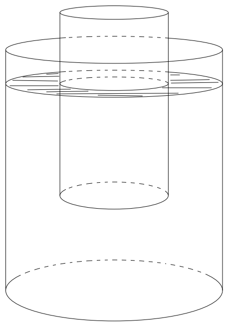
# 5 Lecture examples

You don't have to do these problems, but it will be helpful if you think about them.

**L.1** Moments of inertia.

Without evaluating any integrals, rank the following objects in order of decreasing moment of inertia about their axis of symmetry: (1) a thin ring, (2) a thin disc. All objects have the same mass and radius and are uniform.

Which rolls down a hill faster: a big hoop, or a small flat disc?

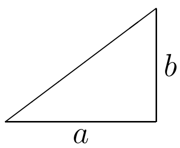


**L.2** Bob. A cylindrical plastic bottle containing some heavy sand bobs up and down as it floats vertically in a cylindrical bucket of water. Estimate the frequency of the bobbing.

**L.3** Clock.

- How does the speed of a pendulum clock depend on the amplitude of its pendulum swing? (Estimate the change in timekeeping if the amplitude changes from $2^{\circ}$ to $10^{\circ}$.)
- The tick-tock of a clock is the sound of small kicks being given to the pendulum, to keep it going. Assuming that each kick is delivered for a very short duration, at what point in the swing should the kicks be given in order to minimize the effect of the kicks on the speed of the clock? Bear in mind that the amplitude of the pendulum's swing may vary from day to day.
- If a clock that is being used for navigation (to determine longitude, by comparing the times of Greenwich midday and local midday) loses a couple of minutes over the course of a voyage, how far off course will the sailors find themselves?

**L.4** Slinky. Use dimensional analysis to predict how the end-to-end travel time of a pulse along a slinky depends on how stretched the slinky is.




**L.5** Triangle. Use dimensional analysis to predict

(a) how the area $A$ of a right-angled triangle depends on the length of its sides $a$ and $b$;

(b) how the squared length of the hypotenuse, $c^2$, depends on $a$ and $b$.

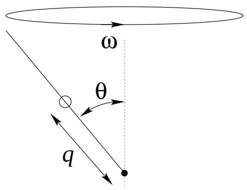
**L.6** Inverse-square orbits. A point mass with cylindrical coordinates $(r, \theta)$ moves on a plane in a circularly-symmetric potential

$$V(r) = -\frac{A}{r}.$$

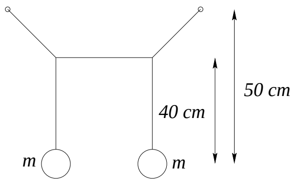
[This is called 'inverse-square' because the force varies as $1/r^2$.] Use conservation of energy $E$ and angular momentum $J$ to find the effective potential for $r$, and derive the equation of motion for $r$. Find the condition that $r$ must satisfy, for a given $J$, for the orbit to be circular, and find the frequency of the small oscillations in $r$ that occur if the particle, initially in a circular orbit, is given a small *radial* kick. Compare this frequency to the original orbital frequency $\dot{\theta}$ and sketch the trajectory of the particle, post-kick.

Repeat the calculations for the case of not-quite-inverse-square potential

$$V(r) = -\frac{A}{r^{1+\alpha}}.$$




**L.7** Bead on wire. A bead can slide without friction on a straight wire. The wire is spun at angular velocity $\omega$. The angle $\theta$ from vertical is fixed. What motions are possible? Solve for the motion of the particle using Lagrangian methods.




**L.8** Coupled pendula. Two bobs are suspended on light threads as shown. What are the normal modes? Estimate their frequencies. Describe what happens when one bob is displaced a little out of the page and released. Estimate the time taken for the energy to move to and fro between the bobs.

**L.9** Cross section. What does the 'cross section' of a target mean? (For example, the cross section of a nucleus that scatters incident particles.)

Defining the cross section to be an area, use dimensional analysis to predict what the dependence of the cross section on the incident particle's mass and velocity is, if the nucleus is modelled as an uncharged hard sphere of radius $a$.

Use dimensional analysis to predict what the dependence of the cross section on the incident particle's mass and velocity is, if the nucleus is not modelled by a hard sphere, but instead, the force exerted by the nucleus is assumed to follow an inverse-square law. (Which is a law of the form $F = A/r^2$, where $A$ is a dimensional constant relating force to distance squared).

**L.10** Ride. In an amusement park, a cylindrical room is spun around a vertical axis and the floor is removed, once everyone is stuck to the walls. Draw all the physical forces acting on an amused occupant. Estimate the speed of rotation. Describe what happens, from the point of view of (i) the occupants, and (ii) an external observer, if an occupant tries to throw a ball radially across the room, (a) without taking into account the rotation; (b) at an angle so that the ball's true velocity is radial.

## Numbers suitable for use on backs of envelopes

(to one decimal place)

| | | |
|---|---|---|
| PROPERTIES OF MATTER | | |
| Weight of an apple | | 1 N |
| Speed of sound | | $300\,\mathrm{m\,s^{-1}}$ |
| Heat capacity of water | | $4000\,\mathrm{J\,kg^{-1}\,K^{-1}} = 1\,\mathrm{cal\,g^{-1}\,K^{-1}}$ |
| Density of air | | $1\,\mathrm{kg/m^3}$ |
| Density of water | | $1000\,\mathrm{kg/m^3} = 1\,\mathrm{g/cm^3}$ |
| Atomic radius | | $10^{-10}\,\mathrm{m}$ |
| Atomic energies | | If $T = 10,000\,\mathrm{K}$, $\mathrm{k}T = 1\,\mathrm{eV}$ |
| Visible light | | from 2 to 4 eV |
| Ionization energy of hydrogen atom | | 14 eV |
| Young's modulus of steel | | $2 \times 10^{11}\,\mathrm{Pa}$ |
| Atmospheric pressure | | $10^5\,\mathrm{Pa}$ |
| CONVERSION FACTORS | | |
| One radian | | $1\ \mathrm{rad} = 60^{\mathrm{o}}$ |
| One day | | $10^5\,\mathrm{s}$ |
| One year | | $\pi \times 10^7\,\mathrm{s}$ |
| | | ($\pi$ seconds is a nanocentury) |
| Speed | | 1 mile per hour $= 0.5\,\mathrm{m\,s^{-1}}$ |
| EARTH | | |
| Radius of earth | | $6 \times 10^6\,\mathrm{m}$ |
| One quarter earth circumference | | $10^7\,\mathrm{m}$ |
| Rotation speed of earth at equator | | $500\,\mathrm{m\,s^{-1}}$ = 1000 miles per hour |
| CONSTANTS | | |
| Light speed | $c$ | $3 \times 10^8\,\mathrm{m\,s^{-1}}$ = 1 foot per ns |
| One mole | $N_A$ | $6 \times 10^{23}$ |
| Molar volume | | $0.02\ \mathrm{m^3} = 20\ \mathrm{l}$ |
| Electron charge | $e$ | $1.6 \times 10^{-19}\,\mathrm{C}$ |
| Nucleon mass | $m_p$ | $1.7 \times 10^{-27}\,\mathrm{kg}$ |

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**Version 3.1**. Feedback is very welcome. You can reach the `metafaq` feedback and question-answering system via `http://wol.ra.phy.cam.ac.uk/teaching/dynamics/`.