Hamiltonian Dynamics Recipe

Before starting: Obtain the Hamiltonian from the Lagrangian.

H will typically be a function of the coordinates $\mathbf{q} = \{q_i\}$ and the velocities $\dot{\mathbf{q}} = \{\dot{q}_i\}$, $H(\mathbf{q}, \dot{\mathbf{q}})$. [It could be time-dependent too.]

Step 1: Substitute for the velocities $\dot{q}_i$ in terms of the conjugate momenta $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$.

This yields the Hamiltonian as a function of $\mathbf{q}$ and $\mathbf{p}$ [and possibly time].

Step 2: We find the partial derivatives of $H(\mathbf{q}, \mathbf{p})$ with respect to its $2N$ arguments. The equations of motion are, for $i = 1 \dots N$:

$$\begin{aligned} \text{I}: \quad \frac{d}{dt} q_i &= \frac{\partial H}{\partial p_i} \\ \text{II}: \quad \frac{d}{dt} p_i &= -\frac{\partial H}{\partial q_i} \end{aligned}$$

Notice the near-symmetry of these equations.

Example:

Motion in a radially symmetric potential.

$$H\left((r,\theta),(\dot{r},\dot{\theta})\right) = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + V(r)$$

$$\begin{aligned} p_r &= m\dot{r} & \Rightarrow \quad \dot{r} &= p_r/m \\ p_\theta &= mr^2\dot{\theta} & \Rightarrow \quad \dot{\theta} &= p_\theta/(mr^2) \end{aligned}$$

$$\Rightarrow \; H\left((r,\theta),(p_r,p_\theta)\right) = \frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}\frac{p_\theta^2}{mr^2} + V(r)$$

$$\begin{aligned} \text{I}: \quad \frac{d}{dt} r &= \frac{\partial H}{\partial p_r} &&= \frac{p_r}{m} \\ \frac{d}{dt}\theta &= \frac{\partial H}{\partial p_\theta} &&= \frac{p_\theta}{mr^2} \\ \text{II}: \quad \frac{d}{dt} p_r &= -\frac{\partial H}{\partial r} &&= \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \\ \frac{d}{dt} p_\theta &= -\frac{\partial H}{\partial \theta} &&= 0 \end{aligned}$$

Comments:

1. Hamilton's equations can be derived from Lagrangian dynamics. [See HF 179, KB 222.]
2. Whereas the Euler-Lagrange equations for a system with N degrees of freedom consist of N *second-order* differential equations for the N functions $\{q_i(t)\}$, Hamilton's equations give $2N$ *first-order* differential equations for the $2N$ functions $\{q_i(t)\}, \{p_i(t)\}$.
3. Because Hamilton's equations are first-order, and because of the symmetry between $\mathbf{q}$ and $\mathbf{p}$, the Hamiltonian formulation may be easier to simulate numerically.
4. If the state-space is defined in terms of the variables $\{q_i\}, \{p_i\}$, it can be proved (see HF 184, 202; KB 230) that *Hamiltonian dynamics conserve state-space volume.* This result is known as *Liouville's theorem.* The motion of the state in the state-space is like the flow of an incompressible fluid.
5. Hamiltonian dynamics are a foundation for quantum mechanics.