Optional reading

In the final lecture we will demonstrate the surprising fact that an inverted pendulum can be made stable by oscillating its support up and down vertically.

Inverted pendulum driven by square wave

You can find sophisticated analyses of the inverted pendulum in various books, *e.g.*, Acheson 'From Calculus to Chaos' (O.U.P.) p. 168, or Hand and Finch p. 395. For a sinusoidal driving motion of amplitude a and frequency ω, the inverted position is shown to be stable if

$$a\omega > \sqrt{2gl}. \tag{1}$$

The left hand side, $a\omega$, is the maximum velocity of the driven support; the right hand side is the velocity of the pendulum tip if it were to fall from vertical to horizontal.

Here we make a cheap and cheerful solution assuming a square-wave driving motion.

Solution for small displacements from vertical

Let the inverted pendulum with length l have its mass concentrated at the tip. Define x to be the angle of displacement. The pivot has a velocity that is square-wave, $\pm v$, and of period $2t$. The amplitude of the pivot motion is $a = vt/2$.

Define $G = g/l$ and $p = v/l$. The quantity p measures the size of the impulse each half-cycle, and can be compared with Gt, which is the (rescaled) impulse delivered by gravity in time t. (Both quantities have been rescaled by $1/(lm)$.) Assuming a small time step t, the equations of motion are

1. during free fall of duration t,

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix} \rightarrow \begin{bmatrix} x + t\dot{x} \\ \dot{x} + Gtx \end{bmatrix} \tag{2}$$

2. during the impulse of size p,

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix} \rightarrow \begin{bmatrix} x \\ \dot{x} \pm px \end{bmatrix} \tag{3}$$

Notice that both these maps are linear in x and $\dot{x}$. And they are both volume-preserving, to order t^2; real Newtonian dynamics are perfectly volume-preserving.

We concatenate an impulse, a free fall, a second opposite impulse, and a second free fall to obtain the map created by one period of the square wave.

Straightforward manipulations show that the one-period map is

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix} \rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{4}$$

where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1+Gt^2+pt-p^2t^2 & 2t-pt^2 \\ 2Gt+Gt^2p-tp^2 & 1+Gt^2-pt \end{bmatrix}. \tag{5}$$

The stability of this map depends on the eigenvalues of the matrix. The matrix has determinant 1, so the only possibilities are for the two eigenvalues to be a complex pair, in which case the orbits of the map are closed ellipses, or they are real and unequal, meaning that the dynamics stretch in one direction and shrink in another, that is, are unstable.

The eigenvalues are given by

$$\lambda = \frac{1}{2}\left[T \pm \sqrt{T^2 - 4\Delta}\right], \tag{6}$$

where T and Δ are the trace and determinant

$$T = A + D, \quad \Delta = AD - BC. \tag{7}$$

Since we know $D = 1$, we can find the condition for stability from the square root: we require

$$T^2 < 4. \tag{8}$$

Now, $T = 2 + 2Gt^2 - p^2t^2$ so we have stability if

$$p > \sqrt{2G} \tag{9}$$

or, equivalently,

$$v > \sqrt{2gl}, \tag{10}$$

which is identical to the condition for stability when we use a sinusoid of amplitude a and frequency ω (as given in Acheson's book),

$$a\omega > \sqrt{2gl}. \tag{11}$$

Discussion

1. The analysis is linear. It predicts that displacements from vertical of all amplitudes are either stable or unstable.

2. Using a square wave doesn't enhance the effect, sadly. I'd expected that a square wave might work better than a pure sinusoid, since a square wave is a sum of a load of sinusoids of higher frequencies, and the higher the frequency, the better, for sinusoids. Why is this additive viewpoint incorrect? What if we drive the pivot with a sum of two velocity profiles? What happens if, say, we gradually increase the amount of some higher frequency?