Solution to L6:

Near-Inverse-square orbits. A point mass with cylindrical coordinates (r, θ) moves on a plane in a circularly-symmetric potential

$$V(r) = -\frac{A}{r^{1+\alpha}}.$$

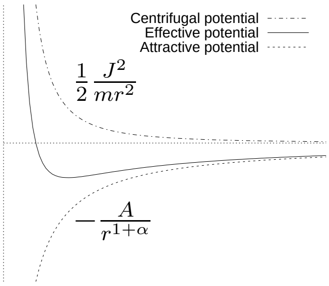
The energy, $E = T + V$, is

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) \quad (1)$$

Substitute for $\dot{\theta}$ using angular momentum

$$J = mr^2\dot{\theta} = \text{constant} \quad (2)$$

and define



$$T_{\text{eff}} = \frac{1}{2}m\dot{r}^2; \quad (3)$$

$$V_{\text{eff}} = \frac{1}{2}\frac{J^2}{mr^2} - \frac{A}{r^{1+\alpha}}. \quad (4)$$

By the energy method, the equation of motion is:

$$m\ddot{r} = -\frac{\partial V_{\text{eff}}}{\partial r} = \frac{J^2}{mr^3} - (1+\alpha)\frac{A}{r^{2+\alpha}}. \quad (5)$$

Circular orbits occur where $\frac{\partial V_{\text{eff}}}{\partial r} = 0$.

$$\frac{\partial V_{\text{eff}}}{\partial r} = -\frac{J^2}{mr^3} + (1+\alpha)\frac{A}{r^{2+\alpha}} \quad (6)$$

This derivative is zero at $r = r_0$, which satisfies:

$$\frac{J^2}{mr_0^3} = (1+\alpha)\frac{A}{r_0^{2+\alpha}}. \quad (7)$$

For small deviations from r_0, there is simple harmonic motion about r_0 with frequency given by 'equation zero',

$$\omega^2_{\text{SHM}} = \frac{1}{m}\left.\frac{\partial^2 V_{\text{eff}}}{\partial r^2}\right|_{r=r_0}. \quad (8)$$

Note use of hygienic differentiation trick.

Pull out a factor of $1/r^3$ so that what's left is easier to differentiate. At (11) we don't need to bother evaluating the derivative of $1/r^3$ because it is multiplied by a quantity* that we know from (7) is zero when $r = r_0$.

At (13) we have substituted for A in terms of J using (7).

We find the 'spring constant', the second derivative of V:

$$\left.\frac{\partial^2 V_{\text{eff}}}{\partial r^2}\right|_{r=r_0} = \frac{\partial}{\partial r}\left[-\frac{J^2}{mr^3} + (1+\alpha)\frac{A}{r^{2+\alpha}}\right]_{r=r_0} \quad (9)$$

$$= \frac{\partial}{\partial r}\left[\left(\frac{1}{r^3}\right)\left(-\frac{J^2}{m} + (1+\alpha)Ar^{1-\alpha}\right)\right]_{r=r_0} \quad (10)$$

$$= \left[\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\right]\left.\left(-\frac{J^2}{m} + (1+\alpha)Ar^{1-\alpha}\right)^*\right|_{r_0} + \left.\frac{1}{r^3}(1+\alpha)(1-\alpha)r^{-\alpha}A\right|_{r_0} \quad (11)$$

$$= 0 + \frac{1}{r_0^{3+\alpha}}(1-\alpha^2)A \quad (12)$$

$$= (1-\alpha)\frac{J^2}{mr_0^4} \quad (13)$$

So the frequency of simple harmonic motion is

$$\omega_{\mathrm{SHM}}^2 = \frac{1}{m}\left.\frac{\partial^2 V_{\mathrm{eff}}}{\partial r^2}\right|_{r=r_0} = (1-\alpha)\frac{J^2}{m^2 r_0^4} = (1-\alpha)\dot{\theta}^2 \tag{14}$$

(using (2)), where $\dot{\theta}$ is the angular velocity of the original circular orbit.

So, if $\alpha > 0$, the radial oscillations have frequency ω_{SHM} that is slightly *smaller* than $\dot{\theta}$.

$$\omega_{\mathrm{SHM}} \simeq (1-\alpha/2)\dot{\theta} \tag{15}$$

So the orbit, which is roughly elliptical, precesses, with the orientation of the ellipse advancing in the same direction as $\dot{\theta}$.

What is the precession rate?

[Let's assume $\alpha > 0$, here; the details are a little different for $\alpha < 0$.]

If it takes N orbits for one complete precession to occur, then in those N orbits, each having period $2\pi/\dot{\theta}$, there must have been $N-1$ of the radial oscillations, each having period $2\pi/\omega_{\mathrm{SHM}}$.

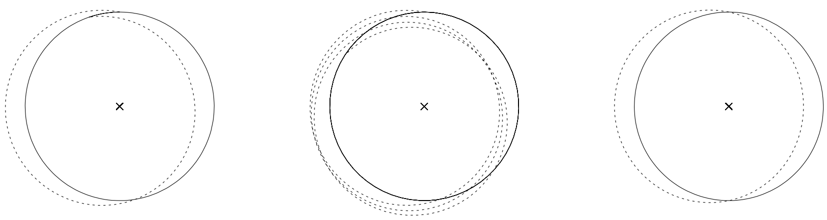
Setting those two times equal,

$$\begin{aligned}
(N-1)\frac{2\pi}{\omega_{\mathrm{SHM}}} &= N\frac{2\pi}{\dot{\theta}} && (16)\\
\Rightarrow\quad 1-\frac{1}{N} &= \frac{\omega_{\mathrm{SHM}}}{\dot{\theta}} && (17)\\
\Rightarrow\quad \frac{1}{N} &= \frac{\alpha}{2}\ \text{(using (15)).} && (18)
\end{aligned}$$

So

$$N = \frac{2}{\alpha} \tag{19}$$

is the number of orbits for precession through 2π.



(a): $\alpha = 0.1$ 1 period | (b): $\alpha = 0.1$ 3 periods | (c): $\alpha = 0$

Figures (a) and (b) show sketches of the precessing orbit for $\alpha = 0.1$ after (a) one period of the radial oscillation; (b) three periods of radial oscillation. Solid line is the circular orbit, and dashed line is the non-circular orbit. Figure (c) shows a sketch of the perturbed orbit for the perfect inverse-square force $\alpha = 0$.