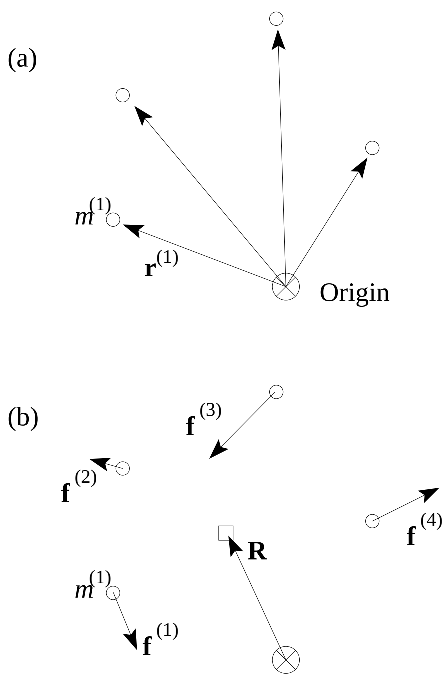
Newtonian Dynamics Revision

Consider a collection of N point masses $m^{(n)}$ moving under the influence of internal and external forces. These masses might make up an extended body, for example. We won't specify the details of the internal forces between particles or the external forces acting. The location of the nth particle is $\mathbf{r}^{(n)}$, and its velocity is $\dot{\mathbf{r}}^{(n)}$. The net force acting on particle n is $\mathbf{f}^{(n)}$. By Newton's second law,

$$m^{(n)}\ddot{\mathbf{r}}^{(n)} = \mathbf{f}^{(n)}. \quad (1)$$

What can we deduce from this?



Definitions

The total mass of the collection of particles is

$$M \equiv \sum_n m^{(n)}. \quad (2)$$

The centre of mass is located at

$$\mathbf{R} \equiv \frac{1}{M}\sum_n m^{(n)}\mathbf{r}^{(n)}. \quad (3)$$

The total momentum is

$$\mathbf{P} \equiv \sum_n m^{(n)}\dot{\mathbf{r}}^{(n)}. \quad (4)$$

The total angular momentum about the origin is

$$\mathbf{J}_O \equiv \sum_n \mathbf{r}^{(n)} \times \left(m^{(n)}\dot{\mathbf{r}}^{(n)}\right). \quad (5)$$

The suffix 'O' is used to show that the angular momentum is defined *with respect to an origin.* Any origin can be chosen (with each $\mathbf{r}^{(n)}$ being mapped to $\mathbf{r}^{(n)} - \mathbf{r}_{O'}$ when the origin is moved to $\mathbf{r}_{O'}$). In general, the value of the angular momentum depends on the origin chosen.

Exercise: Show that the angular momentum is only independent of the origin if the total momentum is zero.

The total angular momentum about the centre of mass is

$$\mathbf{J}_{\mathrm{CM}} \equiv \sum_n m^{(n)}(\mathbf{r}^{(n)} - \mathbf{R}) \times \dot{\mathbf{r}}^{(n)}. \quad (6)$$

The total force acting is

$$\mathbf{F} \equiv \sum_n \mathbf{f}^{(n)}. \quad (7)$$

The total couple acting about the origin is

$$\mathbf{G}_O \equiv \sum_n \mathbf{r}^{(n)} \times \mathbf{f}^{(n)}. \quad (8)$$

Results

The following results are true for any collection of particles.

1. The acceleration of the centre of mass, $\ddot{\mathbf{R}}$, is given by
$$\mathbf{F} = M\ddot{\mathbf{R}}. \tag{9}$$
So the centre of mass moves in just the same way as would a single particle of mass M, subjected to the sum of all the forces.

2. The angular momentum about the origin changes at rate
$$\dot{\mathbf{J}}_O = \mathbf{G}_O. \tag{10}$$

3. The angular momentum about the centre of mass changes at rate
$$\dot{\mathbf{J}}_{\rm CM} = \mathbf{G}_{\rm CM}, \tag{11}$$
where $\mathbf{G}_{\rm CM}$ is the total couple about the centre of mass,
$$\mathbf{G}_{\rm CM} \equiv \sum_n (\mathbf{r}^{(n)} - \mathbf{R}) \times \mathbf{f}^{(n)}. \tag{12}$$

Proofs

We prove the first two results and leave the third as an exercise.

1. We start from Newton's second law (1)
$$m^{(n)}\ddot{\mathbf{r}}^{(n)} = \mathbf{f}^{(n)}.$$
We sum over n.
$$\sum_n m^{(n)}\ddot{\mathbf{r}}^{(n)} = \sum_n \mathbf{f}^{(n)}. \tag{13}$$
The right hand side is $\mathbf{F}$; the left hand side is $M\ddot{\mathbf{R}}$, since
$$M\ddot{\mathbf{R}} = \frac{d^2}{dt^2}\sum_n m^{(n)}\mathbf{r}^{(n)} = \sum_n m^{(n)}\frac{d^2}{dt^2}\mathbf{r}^{(n)}. \tag{14}$$
QED.

2. We start from Newton's second law (1)
$$m^{(n)}\ddot{\mathbf{r}}^{(n)} = \mathbf{f}^{(n)}.$$
We left-multiply by $\mathbf{r}^{(n)}$, using the cross product, and sum over n.
$$\sum_n \mathbf{r}^{(n)} \times m^{(n)}\ddot{\mathbf{r}}^{(n)} = \sum_n \mathbf{r}^{(n)} \times \mathbf{f}^{(n)}. \tag{15}$$
The right hand side is $\mathbf{G}_O$; the left hand side is $\dot{\mathbf{J}}_O$, as we now show.
$$\begin{aligned}\dot{\mathbf{J}}_O &= \frac{d}{dt}\sum_n m^{(n)}\mathbf{r}^{(n)} \times \dot{\mathbf{r}}^{(n)} && (16)\\ &= \sum_n \left[m^{(n)}\dot{\mathbf{r}}^{(n)} \times \dot{\mathbf{r}}^{(n)} + m^{(n)}\mathbf{r}^{(n)} \times \ddot{\mathbf{r}}^{(n)}\right] \\ &= \sum_n \left[0 + m^{(n)}\mathbf{r}^{(n)} \times \ddot{\mathbf{r}}^{(n)}\right] && (17)\end{aligned}$$
QED.

In both proofs, the key idea is that differentiation is a linear operation, so we can interchange the order of the summation and differentiation, as in equation (14).

Further results

If the force on a particle, $\mathbf{f}^{(n)}$, can be divided into two parts,
$$\mathbf{f}^{(n)} = \mathbf{f}^{(n)}_{\rm ext} + \mathbf{f}^{(n)}_{\rm int}, \tag{18}$$
where $\mathbf{f}^{(n)}_{\rm ext}$ is the net 'external' force acting and $\mathbf{f}^{(n)}_{\rm int}$ is the net 'internal' force, and if the internal forces all arise from pairs of forces that obey Newton's 3rd law, *i.e.*, the force exerted on particle n by particle n' is equal and opposite to the force exerted on particle n' by particle n, then the internal forces can be ignored when computing the acceleration of the centre of mass, $\ddot{\mathbf{R}}$ (equation 9), and the rate of change of angular momentum (equation 10). In the total force acting, $\mathbf{F} = \sum_n \mathbf{f}^{(n)}$, all the internal forces cancel each other out. In the total couple about the origin, $\mathbf{G}_O = \sum_n \mathbf{r}^{(n)} \times \mathbf{f}^{(n)}$, all the contributions from internal forces cancel.

So $\mathbf{F}$ is simply the sum of all the external forces, and $\mathbf{G}_O$ is the sum of all the couples of the external forces.

Remember!

A body can have angular momentum about the origin without any *rotation* being involved. When a Northbound plane flies over you, it has got angular momentum about you; its angular momentum vector is directed West.

DJCM. October 11, 2001