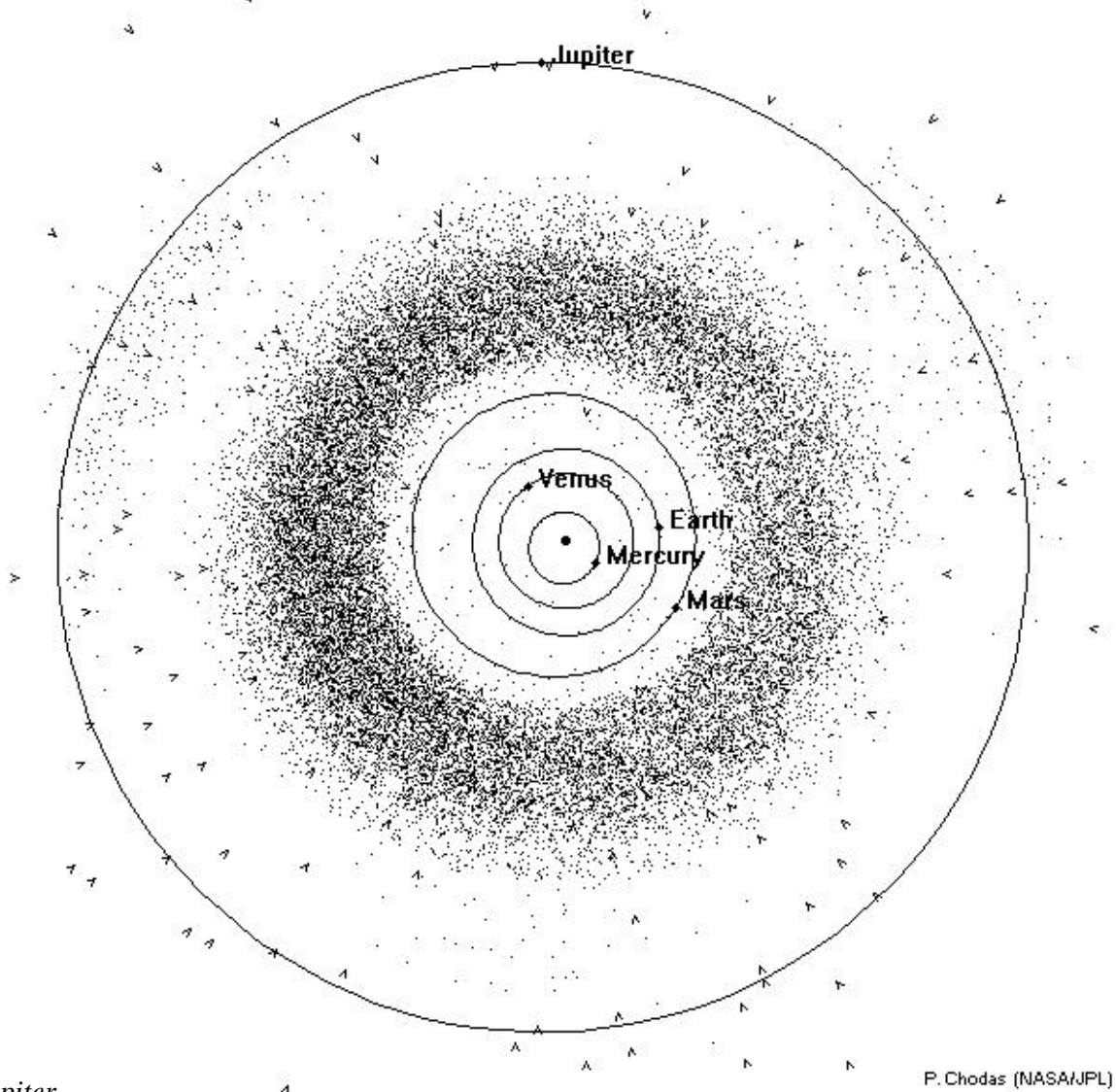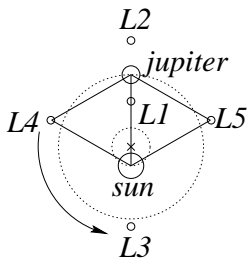Jupiter

Venus
Earth
Mercury
Mars

P. Chodas (NASA/JPL)

L2
jupiter
L4   L1   L5
sun
L3

THE INNER PLANETS

Mercury and Mars have the most eccentric orbits. The wedges are comets and the dots are asteroids. Notice the Trojan asteroids that hang out near Lagrange points L4 and L5 – the points that form equilateral triangles with Jupiter and the Sun.

Solar system figure stolen from `http://ssd.jpl.nasa.gov/`.

| KEPLER'S LAWS	
First Law	All planetary orbits are ellipses, with the Sun at one focus.
Second Law	The line joining a planet to the Sun sweeps out equal areas in equal times.
Third Law	The square of the period of a planet is proportional to the cube of its 'mean distance' to the Sun (the semi-major axis of the ellipse).

Kepler's Ellipses

The solid line is an ellipse

$$r(\theta) = \frac{l}{1 - \epsilon \cos \theta}$$

of eccentricity $\epsilon = 0.1$. [This is the eccentricity of Mars's orbit.]

The dotted line shows a circle fitted to the ellipse. Brahe's data and Kepler's data-fitting were good enough that Kepler was able to reject the circle hypothesis.

Sun
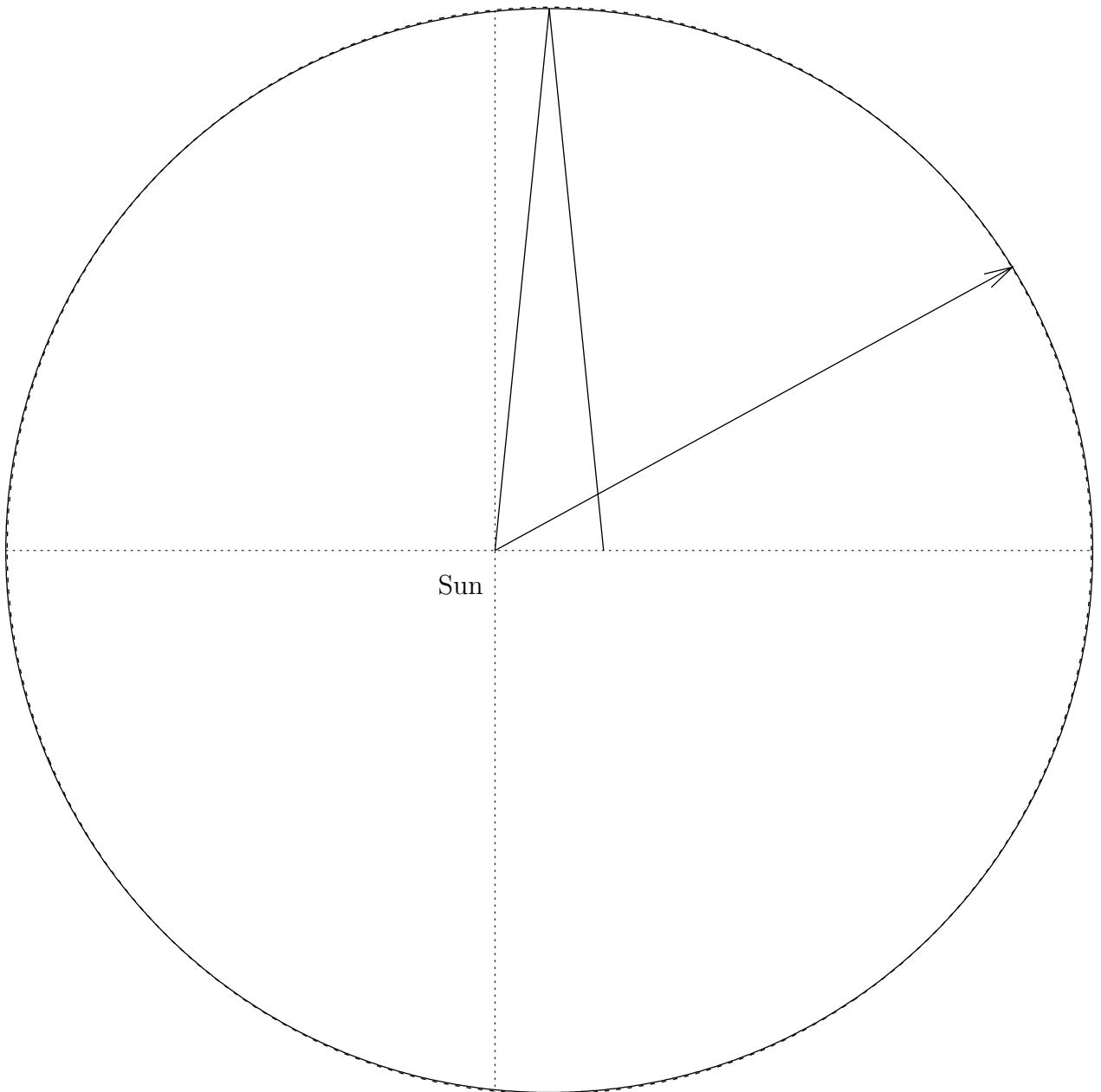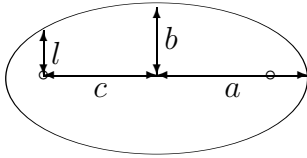
$$r(\theta) \;=\; \frac{l}{1 - \epsilon \cos\theta} \tag{1}$$

Using the property of an ellipse that the sum of the distances to the foci is a constant, and equation (1), we can work out the semi-major axis a, the semi-minor axis b, and the distance from the focus to the centre of the ellipse, c.

$$\text{semi-latus rectum} \quad l \tag{2}$$

$$\text{perihelion} \quad r_{\min} \;=\; \frac{l}{1 + \epsilon} \tag{3}$$

$$\text{aphelion} \quad r_{\max} \;=\; \frac{l}{1 - \epsilon} \tag{4}$$

$$\text{semi-major axis} \quad a \;=\; \frac{l}{1 - \epsilon^2} \tag{5}$$

$$\text{semi-minor axis} \quad b \;=\; \frac{l}{\sqrt{1 - \epsilon^2}} \tag{6}$$

$$c \;=\; \frac{l\epsilon}{1 - \epsilon^2} \tag{7}$$

Because the ellipse is a stretched circle, we can write down its area:

$$A = \pi a b = \pi \frac{l^2}{(1 - \epsilon^2)^{3/2}} = \pi a^2 \sqrt{1 - \epsilon^2}. \tag{8}$$

[You are not expected to learn any of the above relationships.]

Orbital dynamics

Most of the key properties of orbits can be deduced from the energy method (handout 4). In the following, M is the mass of the central body.

ESCAPE VELOCITY

To find the minimum velocity needed to escape from R to infinity, we equate the kinetic energy to the change in potential energy:

$$\frac{1}{2}v_{\mathrm{e}}^2 = \frac{GM}{R} \;\Rightarrow\; v_{\mathrm{e}}^2 = \frac{2GM}{R}. \tag{9}$$

This result holds regardless of the direction that the velocity is pointing. At the surface of earth, where $g = GM/R^2$, the escape velocity is given by $v_{\mathrm{e}} = \sqrt{2gR}$, about $10^4\,\mathrm{ms}^{-1}$ or 20,000 miles per hour.

CIRCULAR ORBITS

In a circular orbit, the velocity and orbital radius R are related by

$$v^2 = \frac{GM}{R}. \tag{10}$$

The escape velocity is $\sqrt{2}$ times the orbital velocity. The total energy E, the kinetic energy T, and the potential energy V are related by

$$V = -2T; \quad E = -T. \tag{11}$$

This relationship has a curious implication. If the satellite loses energy under friction, and remains in a circular orbit, the radius of the orbit decreases and *the satellite speeds up*.

ELLIPTICAL ORBITS

The period of an elliptical orbit T and the energy of the orbit E are both simply related to the *semi-major axis* of the ellipse, a.

$$E = -\frac{GMm}{2a} \tag{12}$$

$$T^2 = 4\pi^2 \frac{a^3}{GM} \tag{13}$$

The angular momentum J is related to the semi-latus rectum, l.

$$J = m\sqrt{GMl} \tag{14}$$

That this is so, at least for small deviations from circular orbits, can be seen by recalling the energy method result: a small radial kick (which does not change the angular momentum) leads to an elliptical orbit with identical semi-latus rectum.

ORBITAL TRANSFER

Assume we wish to get from earth's orbit $(r = r_{\mathrm{E}})$ to Jupiter's $(r = r_{\mathrm{J}})$ by making a single impulse at the start. How big an impulse is required?

Post-impulse we are on an elliptical orbit with $r_{\min} = r_{\mathrm{E}}$ and $r_{\max} = r_{\mathrm{J}}$. If we can find the angular momentum J of such an orbit, we can deduce the impulse required. The effective potentials are equal at the two points $r_{\min} = r_{\mathrm{E}}$ and $r_{\max} = r_{\mathrm{J}}$:
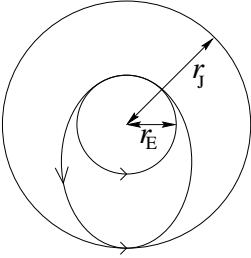
$$E = \frac{1}{2}\frac{J^2}{mr_{\mathrm{E}}^2} - \frac{GMm}{r_{\mathrm{E}}} = \frac{1}{2}\frac{J^2}{mr_{\mathrm{J}}^2} - \frac{GMm}{r_{\mathrm{J}}}. \tag{15}$$

Rearranging all the terms in J onto one side,

$$\frac{J^2}{2m}\left(\frac{1}{r_{\mathrm{E}}^2} - \frac{1}{r_{\mathrm{J}}^2}\right) = GMm\left(\frac{1}{r_{\mathrm{E}}} - \frac{1}{r_{\mathrm{J}}}\right) \tag{16}$$

$$\frac{J^2}{2m}\left(\frac{1}{r_{\mathrm{E}}} - \frac{1}{r_{\mathrm{J}}}\right)\left(\frac{1}{r_{\mathrm{E}}} + \frac{1}{r_{\mathrm{J}}}\right) = GMm\left(\frac{1}{r_{\mathrm{E}}} - \frac{1}{r_{\mathrm{J}}}\right) \tag{17}$$

$$\frac{J^2}{2m} = GMm\left/\left(\frac{1}{r_{\mathrm{E}}} + \frac{1}{r_{\mathrm{J}}}\right)\right. \tag{18}$$

So the new orbit's angular momentum $J_{\mathrm{after}} = J$ is

$$J_{\mathrm{after}} = m\sqrt{GM\frac{2r_{\mathrm{E}}r_{\mathrm{J}}}{r_{\mathrm{E}} + r_{\mathrm{J}}}}. \tag{19}$$

The angular momentum before the impulse was

$$J_{\mathrm{before}} = m\sqrt{GMr_{\mathrm{E}}}. \tag{20}$$

From these we can deduce the impulse:

$$\Delta v = \sqrt{\frac{GM}{r_{\mathrm{E}}}}\left[\sqrt{\frac{r_{\mathrm{J}}}{(r_{\mathrm{E}} + r_{\mathrm{J}})/2}} - 1\right]. \tag{21}$$

Gravitational slingshot

The expression

$$r(\theta) = \frac{l}{1 - \epsilon \cos(\theta - \theta_0)}$$

describes all the possible orbits of a tiny mass near a large mass located at the origin. θ_0 determines the orientation of the orbit. The figure shows orbits with fixed θ_0 and l, and different eccentricities ϵ: $\epsilon = 0, 0.3, 0.7, 1, 1.5$.

|	
$\epsilon = 0$	circle
$\epsilon \in (0, 1)$	ellipse
$\epsilon \in (-1, 0)$	ellipse with $r_{\min}$ and $r_{\max}$ interchanged
$\epsilon = 1$	parabola
$|\epsilon| > 1$	hyperbola

If $|\epsilon| > 1$ then the expression $r(\theta)$ describes two branches, both of which are shown in the above figure. Only one of the branches is a solution of the equation of motion, however. The other branch is the solution of the equation of motion for a *repulsive* $1/r^2$ force.

Slingshot

A spacecraft arrives at Jupiter with velocity parallel to Jupiter's, but smaller. As Jupiter overtakes the spacecraft, we transform into Jupiter's frame.

(a)   (b)
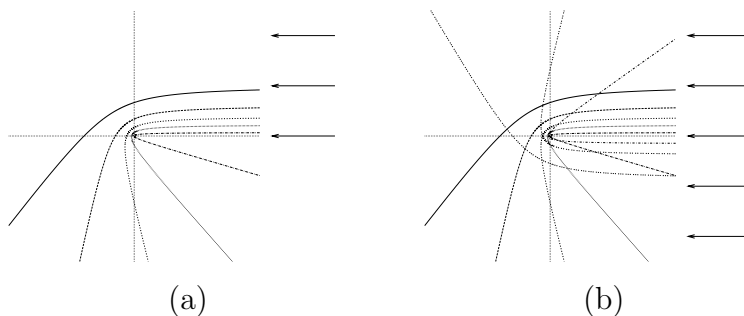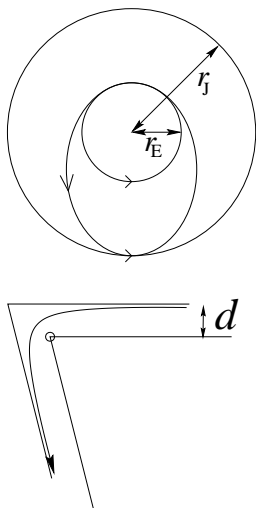
The outcome depends on the 'impact parameter' d. Figure (a) shows possible trajectories (hyperbolas) if the craft arrives on the upper side of Jupiter. Figure (b) shows possible trajectories for arrivals on both sides.

Before   After

v_{before} Δv

v_{J}

v_{after} Δv

v_{J}

(c)
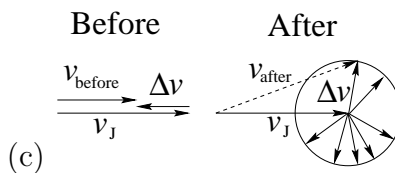
The interaction is an elastic collision. The speed of the spacecraft afterwards, relative to Jupiter, is still Δv, but the direction has changed. Figure (c) shows the velocities before and after in the sun frame (v_{after} is the dashed vector). Notice that whatever orientation is chosen, the speed of the spacecraft in the sun frame is always increased by the collision. In this way, the Voyager spacecraft stole energy from Jupiter to speed themselves on their way to Saturn, Neptune and Uranus. The slingshot manoeuvre was also used by Galileo (Venus, Earth, Earth $\rightarrow$ Jupiter). On arrival, Galileo slowed itself down relative to Jupiter by a slingshot around the moon Io, and a lengthy thrust of its $400\,\text{N}$ engine. It has since adjusted its orbit around Jupiter using further slingshots around other Jovian moons.
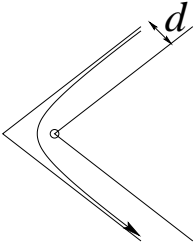
IMPACT PARAMETER RELATIONSHIPS

Page 3 gave formulae relating a, b, c, $r_{\min}$, $r_{\max}$, l and ϵ. The impact parameter d is the distance between the trajectory at infinity and the trajectory for direct impact. In this figure, d is shown in the standard orientation such that $r(\theta) = \frac{l}{1-\epsilon\cos\theta}$ and the point of closest approach is at $\theta = \pi$ (in contrast, on the previous page, all particles arrived from a common direction).

We can relate d to the other parameters using the angular momentum at infinity, $J = mv_\infty d$, and the previously derived relationship between J and l,

$$J^2 = GMm^2l, \tag{22}$$

to obtain

$$l = \frac{v_\infty^2 d^2}{GM} \tag{23}$$

We can eliminate the energy–like terms using

$$E = \frac{1}{2}mv_\infty^2 = -\frac{GMm(1-\epsilon^2)}{2l}, \tag{24}$$

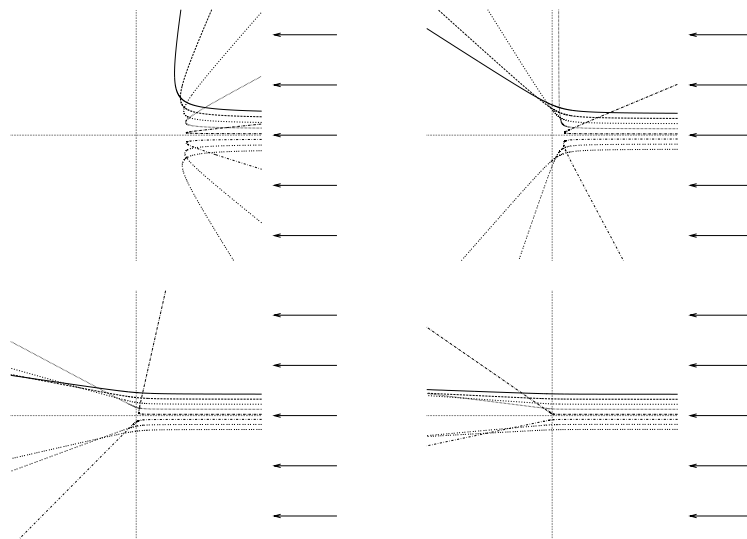giving the convenient relationship

$$d = \frac{l}{\sqrt{\epsilon^2 - 1}}. \tag{25}$$

The impact parameter d of a hyperbola equals i times the semi-minor axis $b = l/\sqrt{1 - \epsilon^2}$. You are not expected to memorize these relationships, but you should understand how they can be derived.

Repulsive inverse–square interactions

If we change the sign on the inverse square force, the only orbits that are possible are hyperbolas. The hyperbola has two branches, one of which is the solution for the attractive force, and the other is the solution for the repulsive force.

The figures below show the trajectories for a variety of impact parameters for incoming velocities equal to 1, 2, 4 and 8 units respectively.

DJCM. November 15, 2001