Strain and Stress

When an elastic body is deformed, internal restoring forces are produced. The **strain** at a point describes the deformation in the vicinity of that point. The **stress** describes the forces that maintain the deformation. In a *linear elastic body*, the stress is linearly related to the strain. Compressions, extensions, and shears are examples of strains. Pressure, tension, and shear stress are examples of stresses.

Summation convention has not been used here; but you'll see, all the same, that wherever there is a repeated index, it is summed over, so the equations would have looked nice and clean with summation convention.

JUNIOR PHYSICS STRAIN AND STRESS

The old definitions of strain and stress which we will generalize are:

Strain ϵ is a fractional extension.

Stress τ is a force per unit area.

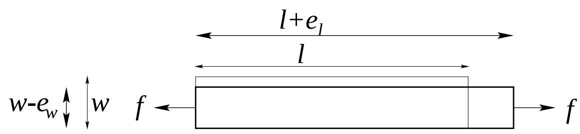


Figure 1: A simple extension

You may have encountered two quantities called the **Young's modulus** Y and the **Poisson ratio** σ of a material. These relate the strain and stress to each other as follows. If a rod of length l, width w, and cross–sectional area A is subjected to a force f and its length increases by e_l and its width *decreases* by e_w, then we define the Young's modulus to be the ratio of the lengthways stress f/A to the lengthways strain e_l/l,

$$Y = \frac{f/A}{e_l/l}; \tag{1}$$

and we define the Poisson ratio to be minus the strain ratio,

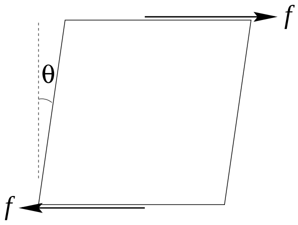
$$\sigma = \frac{e_w/w}{e_l/l}. \tag{2}$$

Strains are dimensionless, so the Young's modulus has the dimensions of a stress. It defines the stress that would produce a strain of 1, *i.e.*, the stress that would double the length of a sample (assuming the material would behave linearly that far!). The Young's modulus of steel is 2×10^{11} Pa.

The Poisson ratio is dimensionless. A typical value for σ is between 0 and 1/2; only values between -1 and 1/2 are physically possible, and negative values are rare — most things shrink widthways when you stretch them lengthways.

You may also have come across the terms **bulk modulus** and **shear modulus**. The bulk modulus B is the ratio of the pressure to the fractional reduction in volume of a sample, when it is subjected to isotropic pressure.

$$B = \frac{p}{-\Delta V/V}. \tag{3}$$



The details of this section are provided as optional reading.

The shear modulus n is the ratio of the shear force per unit area to the shear angle, θ,

$$n = \frac{f/A}{\theta}. \tag{4}$$

WHAT ARE STRAIN AND STRESS?

Strain and stress are not scalars. They both have directional properties. Strain and stress are not vectors either: vectors change sign if we rotate our coordinate system through 180 degrees, but a strain looks identical to itself if we spin through 180 degrees.

Strain and stress are matrices, also known as second-rank tensors.

DEFINITION OF STRAIN

Imagine a deformation of a body such that the point in the body that was at location $\mathbf{x}$ is displaced through $\mathbf{u}(\mathbf{x})$ to $\mathbf{x}+\mathbf{u}(\mathbf{x})$. If the deformation is not a uniform translation or rotation then distances between pairs of points in the body will have changed. The **strain** at a point describes *by how much distances between nearby pairs of points have changed.*

Consider a pair of points $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$ in the undeformed body, where $d\mathbf{x}$ is a small displacement. These are deformed to locations $\mathbf{x}+\mathbf{u}(\mathbf{x})$ and $\mathbf{x}+d\mathbf{x}+\mathbf{u}(\mathbf{x}+d\mathbf{x})$. The squared distance between the deformed points is

$$\begin{aligned} &\sum_i \left(dx_i + \sum_j \frac{\partial u_i}{\partial x_j} dx_j \right)^2 \\ &= \sum_i (dx_i)^2 + 2\sum_{i,j} \left(dx_i \frac{\partial u_i}{\partial x_j} dx_j \right) + \sum_{i,j,j'} \left(\frac{\partial u_i}{\partial x_j} dx_j \frac{\partial u_i}{\partial x_{j'}} dx_{j'} \right) \end{aligned} \tag{5}$$

The first term, $\sum_i dx_i^2$, is the original squared distance between the points; the remaining terms give us the *change* in squared distance between the points, which is what we are interested in.

The change in squared distance, as a function of $d\mathbf{x}$, is a quadratic function of $d\mathbf{x}$. We can therefore manipulate it into quadratic form, and define the matrix in that quadratic form to be the strain tensor.

$$D(d\mathbf{x}) \equiv \sum_{i,j} dx_i \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dx_j + \sum_{i,j,j'} dx_j \left(\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_{j'}} \right) dx_{j'} \tag{6}$$

$$= \sum_{i,j} dx_i \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_k \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right] dx_j \tag{7}$$

We define the strain tensor to be

$$\epsilon_{ij} \equiv \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_k \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right], \tag{8}$$

then the change in squared distance is

$$D(d\mathbf{x}) = 2 \sum_{i,j} dx_i \, \epsilon_{ij} \, dx_j \tag{9}$$

We will usually assume that we are concentrating on *small* strains, *i.e.*, that the derivatives $\frac{\partial u_i}{\partial x_j}$ are all much smaller than 1. This assumption allows us

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to ignore the second term, which is quadratic in $\frac{\partial u_k}{\partial x_j}$, so that *the strain tensor for small distortions* is

$$\epsilon_{ij} \equiv \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]. \tag{10}$$

The stress tensor

The stress is a force per unit area. Force is a vector. Area is also represented by a vector (the normal to the area). So, if

$$\text{force} = \text{stress times area}, \tag{11}$$

stress must in general be a matrix. We denote it by $\underline{\underline{\tau}}$, and define the relationship between force $\mathbf{f}$ and a small area $\mathbf{a}$ to be:

$$\mathbf{f} = \underline{\underline{\tau}}\mathbf{a} \quad \text{or} \quad f_i = \sum_j \tau_{ij} a_j. \tag{12}$$

If we assume that the material is at equilibrium, $\underline{\underline{\tau}}$ must be a symmetric matrix: otherwise (as you can check) there would be a net couple acting on a small cube of material, so it would not be at equilibrium – it would start spinning.

In the special case of isotropic hydrostatic pressure p, the force and the area are in the same direction, so the stress tensor is just

$$\underline{\underline{\tau}} = -p\mathbf{1}, \tag{13}$$

where $\mathbf{1}$ is the identity matrix. Our convention for direction will be that the area vector points outwards from the body, and the force vector is the force exerted on the area by the surroundings. In the case of positive pressure, the force is directed inwards, hence the minus sign in the hydrostatic $\underline{\underline{\tau}}$ (13).

In general, the force exerted on an area is not directed exactly along the normal to the area, and $\underline{\underline{\tau}}$ is more complicated than the hydrostatic $\underline{\underline{\tau}}$ above.

The strain–stress relationship

How are the strain $\underline{\underline{\epsilon}}$ and stress $\underline{\underline{\tau}}$ related to each other? The strain–stress relationship can be quite complicated: try stressing a piece of cotton cloth in various orientations, for example; you'll find that the response to an extension stress varies with the orientation along which it is applied. The response to a shear stress also varies with the orientation of the shear.

The details of this section are provided as optional reading.

A linear elastic material is one in which the stress is a linear function of the strain. The most general linear relationship between two 3×3 matrices is:

$$\tau_{ij} = \sum_{k,l} Y_{ijkl}\, \epsilon_{kl}, \tag{14}$$

where $\{Y_{ijkl} : i, j, k, l = 1, 2, 3\}$ is a collection of $3 \times 3 \times 3 \times 3 = 81$ numbers.

In an isotropic material, however, the relationship is much simpler. An isotropic material is one whose physics looks the same in any orthonormal basis; in particular, the strain–stress relationship (14) is invariant under orthogonal changes of basis. The most general isotropic linear relationship between two symmetric matrices turns out to be

$$\tau_{ij} = \mu\epsilon_{ij} + \nu\delta_{ij} \sum_k \epsilon_{kk}, \tag{15}$$

that is,

$$\underline{\underline{\tau}} = \mu \underline{\underline{\epsilon}} + \nu \mathbf{1} \mathrm{Trace}\, \underline{\underline{\epsilon}}, \tag{16}$$

where μ and ν are two parameters defining the elastic properties of the material. There is a similar inverse relationship of the form

$$\underline{\underline{\epsilon}} = \phi \underline{\underline{\tau}} + \chi \mathbf{1} \mathrm{Trace}\, \underline{\underline{\tau}}. \tag{17}$$

The fact that these linear relationships involve two parameters corresponds to the fact that two parameters (the Young's modulus and the Poisson's ratio) are normally used to characterise an elastic material. A nonisotropic material requires more parameters to describe its elasticity.

This section is provided as optional reading.

THE STRAIN-STRESS RELATIONSHIP IN TERMS OF Y AND σ

We now express the linear relationship (16) in terms of Y and σ.

In the situation depicted in figure 1, the stress is:

$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{18}$$

where $\tau_{11} = f/A$. [The axes have been aligned with the length, width and depth, respectively.] The strain is:

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \frac{1}{Y}\tau_{11} & 0 & 0 \\ 0 & -\sigma\frac{1}{Y}\tau_{11} & 0 \\ 0 & 0 & -\sigma\frac{1}{Y}\tau_{11} \end{bmatrix}. \tag{19}$$

Now we can use this $\underline{\underline{\epsilon}}, \underline{\underline{\tau}}$ pair to deduce the values of ϕ and χ in the linear relationship (17). Let's manipulate (19) into the form of (17).

$$\underline{\underline{\epsilon}} = \begin{bmatrix} (1+\sigma)\frac{1}{Y}\tau_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -\sigma\frac{1}{Y}\tau_{11} & 0 & 0 \\ 0 & -\sigma\frac{1}{Y}\tau_{11} & 0 \\ 0 & 0 & -\sigma\frac{1}{Y}\tau_{11} \end{bmatrix} \tag{20}$$

$$= \frac{(1+\sigma)}{Y}\underline{\underline{\tau}} - \frac{\sigma}{Y}\mathbf{1}\mathrm{Trace}\,(\underline{\underline{\tau}}). \tag{21}$$

This is the only way to relate $\underline{\underline{\epsilon}}$ and $\underline{\underline{\tau}}$ in the form of (17), so it must be the general relationship.

$$\underline{\underline{\epsilon}} = \frac{(1+\sigma)}{Y}\underline{\underline{\tau}} - \frac{\sigma}{Y}\mathbf{1}\mathrm{Trace}\,(\underline{\underline{\tau}}) \tag{22}$$

In many problems, the natural basis will be the eigenvector basis of $\underline{\underline{\epsilon}}$ and $\underline{\underline{\tau}}$, in which case it may be sufficient to memorize the relationship between the diagonal elements, which is:

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$$\epsilon_{11} = \frac{1}{Y}\left[\tau_{11} - \sigma(\tau_{22} + \tau_{33})\right], \tag{23}$$

and two similar equations for ϵ_{22} and ϵ_{33}. This equation is especially easy to derive from the definition of the Young's modulus and the Poisson ratio.

We now solve for μ and ν by inverting this general relationship. We proceed by finding $\mathrm{Trace}\,(\underline{\underline{\epsilon}})$ in terms of $\mathrm{Trace}\,(\underline{\underline{\tau}})$.

$$\mathrm{Trace}\,(\underline{\underline{\epsilon}}) = \frac{(1+\sigma)}{Y}\mathrm{Trace}\,(\underline{\underline{\tau}}) - \frac{\sigma}{Y}\mathrm{Trace}\,(\mathbf{1})\mathrm{Trace}\,(\underline{\underline{\tau}}) \tag{24}$$

Now $\mathrm{Trace}\,(\mathbf{1}) = 3$, so

$$\mathrm{Trace}\,(\underline{\underline{\tau}}) = \frac{Y}{(1-2\sigma)}\mathrm{Trace}\,(\underline{\underline{\epsilon}}). \tag{25}$$

Substituting this into the ϵ–τ relationship (22),

$$\underline{\underline{\epsilon}} = \frac{(1+\sigma)}{Y}\underline{\underline{\tau}} - \frac{\sigma}{(1-2\sigma)}\mathbf{1}\mathrm{Trace}\,(\underline{\underline{\epsilon}}) \tag{26}$$

$$\Rightarrow \underline{\underline{\tau}} = \frac{Y}{(1+\sigma)}\left[\underline{\underline{\epsilon}} + \frac{\sigma}{(1-2\sigma)}\mathbf{1}\mathrm{Trace}\,(\underline{\underline{\epsilon}})\right]. \tag{27}$$

This section is provided as optional reading.

Bulk modulus

We can obtain the bulk modulus and shear modulus from the general linear relationship (22), reproduced here:

$$\underline{\underline{\epsilon}} = \frac{(1+\sigma)}{Y}\underline{\underline{\tau}} - \frac{\sigma}{Y}\mathbf{1}\mathrm{Trace}\,(\underline{\underline{\tau}}). \tag{28}$$

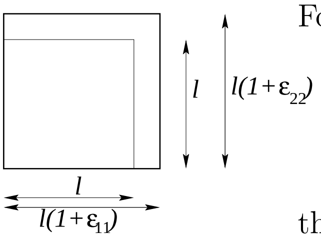
In the case of an isotropic stress,

$$\underline{\underline{\tau}} = -p\mathbf{1}, \tag{29}$$

the strain is also isotropic.

$$\underline{\underline{\epsilon}} = \left[\frac{-p(1+\sigma)}{Y} - \frac{1}{Y}(-3p\sigma)\right]\mathbf{1} \tag{30}$$

$$= -\frac{(1-2\sigma)p}{Y}\mathbf{1}. \tag{31}$$



For a small strain of the form

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}, \tag{32}$$

the fractional change in volume is

$$\frac{\Delta V}{V} = \sum_i \epsilon_{ii}. \tag{33}$$

[In fact, since the right hand side is the trace of $\underline{\underline{\epsilon}}$, which is invariant under orthogonal changes of basis, this expression for the fractional change in volume is correct for any $\underline{\underline{\epsilon}}$.] So from (31), the fractional change in volume is

$$\frac{\Delta V}{V} = -\frac{3(1-2\sigma)p}{Y}, \tag{34}$$

so the bulk modulus is

$$B = \frac{p}{-\Delta V/V} = \frac{Y}{3(1-2\sigma)}. \tag{35}$$

The details of this section are provided as optional reading.

Shear modulus

In a shear experiment as depicted on page 2, the stress is:

$$\underline{\underline{\tau}} = \begin{bmatrix} 0 & \tau_{12} & 0 \\ \tau_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{36}$$

where $\tau_{12} = f/A$. The strain is:

$$\underline{\underline{\epsilon}} = \begin{bmatrix} 0 & \theta/2 & 0 \\ \theta/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

Plugging these into the general linear relationship (22), we can immediately solve for the ratio of τ_{12} to the shear angle θ.

$$\theta/2 = (1+\sigma)\frac{1}{Y}\tau_{12} \tag{38}$$

so the shear modulus is

$$n = \frac{\tau_{12}}{\theta} = \frac{Y}{2(1+\sigma)}. \tag{39}$$

Principal axes

The strain tensor $\underline{\underline{\epsilon}}$ corresponds to a quadratic form, and there is always a choice of orthonormal basis in which $\underline{\underline{\epsilon}}$ is diagonal.

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}, \tag{40}$$

In this basis, the strain is simply an extension or compression along each of the *principal axes*, which are the eigenvectors of $\underline{\underline{\epsilon}}$.

This has the perhaps surprising implication that any strain such as a shear is (locally) just like a set of three perpendicular compressions or extensions.

If we've found the eigenvectors of $\underline{\underline{\epsilon}}$, then we have also found the eigenvectors of $\underline{\underline{\tau}}$, since they are the same, as you can easily confirm from the general linear relationship (22). Thus in this same basis, there are no shear stresses.

In general, the principal axes and their associated eigenvalues will change continuously as a function of location $\mathbf{x}$ in the deformed body.

This section is provided as optional reading.

Potential energy

Just as the potential energy in a simple spring whose extension is x and whose tension is $f = kx$ is $V = \frac{1}{2}fx$, the potential energy per unit volume in an elastic material is

$$V = \frac{1}{2}\sum_{ik} \tau_{ik}\epsilon_{ik} = \frac{1}{2}\text{Trace}\,(\underline{\underline{\tau\epsilon}}). \tag{41}$$

This result is most easily established in the basis aligned with the principal axes.