1B Physics A 2010 (2) A5:

An analogue to digital converter quantizes an input voltage which is randomly distributed, with a standard deviation very much larger than ΔV, the spacing of the digitised levels. Show that the r. m. s. of the difference between the input and the digitised output is $\Delta V/\sqrt{12}$.

In Experimental Methods lecture 8, slide 9, the case of Gaussian errors is considered. I thought I would add some detail to make the effect of the choice of noise model more obvious.

The case considered on the slide is a measured data set $\{x_i, y_i\}_{i=1}^N$ with no error in $\mathbf{x}$ and Gaussian error with known σ_i in $\mathbf{y}$, sampled from a true signal $y(x)$.

Reminder of the Gaussian distribution:

$$\mathcal{N}(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp - \frac{(x-\mu)^2}{2\sigma^2}$$

By using a Gaussian noise model, we are saying that each measured value y_i is a sample from a Gaussian whose mean is the true value $y(x_i)$

$$p(y_i|x_i, y, \sigma_i) = \mathcal{N}(y_i|y(x_i), \sigma_i) \ .$$

Because the Gaussian is just translated by a change in mean, we can express the same thing differently by saying that each measured value y_i is the true value $y(x_i)$ to which Gaussian noise δ_i has been added

$$y_i = y(x_i) + \delta_i$$

or

$$p(y_i - y(x_i)|x_i, y, \sigma_i) = \mathcal{N}(0, \sigma_i) = p(\delta_i|\sigma_i) \ .$$

The difference between the true signal $y(x_i)$ and the measurement y_i is given by δ_i with distribution $\mathcal{N}(0, \sigma_i)$, and the r. m. s. value of the difference for a single measurement is $\sqrt{\langle\delta^2\rangle}$.

$$\begin{aligned}
\langle\delta^2\rangle &= \int_{-\infty}^{\infty} \delta^2\, p(\delta)\,\mathrm{d}\delta \\
&= \int_{-\infty}^{\infty} \delta^2\, \mathcal{N}(0,\sigma)\,\mathrm{d}\delta \\
&= \int_{-\infty}^{\infty} \delta^2\, \frac{1}{\sigma\sqrt{2\pi}} \exp\frac{-\delta^2}{2\sigma^2}\,\mathrm{d}\delta
\end{aligned}$$

Back to our favourite integral from the probability densities of hydrogen, e. g., Quantum Mechanics problem sheet 3, question 35:

$$\begin{aligned}
I_n &= \int_{-\infty}^{\infty} x^n \mathrm{e}^{-\alpha x^2}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} x^{n-1}\, x\mathrm{e}^{-\alpha x^2}\,\mathrm{d}x \\
&= \left[x^{n-1}\frac{\mathrm{e}^{-\alpha x^2}}{-2\alpha}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (n-1)x^{n-2}\,\frac{\mathrm{e}^{-\alpha x^2}}{-2\alpha}\,\mathrm{d}x \\
&= \frac{n-1}{2\alpha} I_{n-2}
\end{aligned}$$

$$
\begin{aligned}
I_0 &= \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha x^2}\,\mathrm{d}x \\
I_0^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} \mathrm{d}r\, r\mathrm{e}^{-\alpha r^2} \\
&= [\theta]_0^{2\pi} \left[\frac{\mathrm{e}^{-\alpha r^2}}{-2\alpha}\right]_0^{\infty} \\
&= \frac{\pi}{\alpha} \\
I_0 &= \sqrt{\frac{\pi}{\alpha}}
\end{aligned}
$$

$I_1 = 0$ as it is the integral of an odd function.

$$
\begin{aligned}
\langle\delta^2\rangle &= \frac{1}{I_0}\frac{2-1}{2\alpha}I_0 \\
&= \sigma^2 \\
\sqrt{\langle\delta^2\rangle} &= \sigma
\end{aligned}
$$

Hopefully you would have expected the r. m. s. value of a variable whose distribution is given by a Gaussian of zero mean and width σ to be σ!

Now for the ADC quantization problem. The effect of quantization is to add an error to the signal whose magnitude is at worst equal to half the spacing $\ell = \Delta V$ of the digitised levels. The simplest consistent noise model is a uniform distribution over the interval $\pm\ell/2$:

$$
p(\delta|\ell) = \begin{cases} 1/\ell & \text{for } |\delta| \le \ell/2 \\ 0 & \text{otherwise.} \end{cases}
$$

$$
\begin{aligned}
\langle\delta^2\rangle &= \int_{-\infty}^{\infty} \delta^2\, p(\delta)\,\mathrm{d}\delta \\
&= \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \frac{\delta^2}{\ell}\,\mathrm{d}\delta \\
&= \frac{1}{\ell}\left[\frac{\delta^3}{3}\right]_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \\
&= \frac{\ell^2}{12} \\
\sqrt{\langle\delta^2\rangle} &= \frac{\ell}{\sqrt{12}}
\end{aligned}
$$

So the r. m. s. of the difference between the input and the digitised output is $\Delta V/\sqrt{12}$.